A MODEL FOR THE QUASI-STATIC GROWTH OF BRITTLE FRACTURES: EXISTENCE AND APPROXIMATION RESULTS

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Abstract. We give a precise mathematical formulation of a variational model for the irreversible quasi-static evolution of brittle fractures proposed by G.A. Francfort and J.-J. Marigo, and based on Griffith's theory of crack growth. In the two-dimensional case we prove an existence result for the quasi-static evolution and show that the total energy is an absolutely continuous function of time, although we can not exclude that the bulk energy and the surface energy may present some jump discontinuities. This existence result is proved by a time discretization process, where at each step a global energy minimization is performed, with the constraint that the new crack contains all cracks formed at the previous time steps. This procedure provides an effective way to approximate the continuous time evolution.

Keywords: variational models, energy minimization, free-discontinuity problems, crack propagation, quasi-static evolution, brittle fractures, Griffith's criterion, stress intensity factor.

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1. INTRODUCTION

Since the pioneering work of A. Griffith [22], the growth of a brittle fracture is considered to be the result of the competition between the energy spent to increase the crack and the corresponding release of bulk energy. This idea is the basis of the celebrated Griffith's criterion for crack growth (see, e.g., [35]), and is used to study the crack propagation along a preassigned path. The actual path followed by the crack is often determined by using different criteria (see, e.g., [17], [35], [36]).

Recently G.A. Francfort and J.-J. Marigo [21] proposed a variational model for the quasi-static growth of brittle fractures, based on Griffith's theory, where the interplay between bulk and surface energy determines also the crack path.

The purpose of this paper is to give a precise mathematical formulation of a variant of this model in the *two-dimensional case*, and to prove an existence result for the *quasi-static evolution of a fracture* by using the *time discretization method* proposed in [21].

To simplify the mathematical description of the model, we consider only *linearly elastic homogeneous isotropic materials*, with Lamé coefficients λ and μ. We restrict our analysis to the case of an *anti-plane shear*, where the reference configuration is an infinite cylinder $\Omega\times\mathbb{R}$, with $\Omega \subset \mathbb{R}^2$, and the displacement has the special form $(0, 0, u(x_1, x_2))$ for every $(x_1, x_2, y) \in \Omega\times\mathbb{R}$. We assume also that the cracks have the form $K\times\mathbb{R}$, where K is a compact set in $\overline{\Omega}$. In this case the notions of bulk energy and surface energy refer to a finite portion of the cylinder determined by two cross sections separated by a unit distance. The *bulk energy* is given by

$$\frac{\mu}{2}\int_{\Omega\setminus K} |\nabla u|^2 dx\,, \tag{1.1}$$

while the *surface energy* is given by

$$k\,\mathcal{H}^1(K)\,, \tag{1.2}$$

where k is a constant which depends on the toughness of the material, and $\mathcal{H}^1$ is the *one-dimensional Hausdorff measure*, which coincides with the ordinary length in case K is a rectifiable arc. For simplicity we take $\mu = 2$ and $k = 1$ in (1.1) and (1.2).

We assume that Ω is a *connected bounded open set* with a *Lipschitz boundary* $\partial\Omega$. As in [21], we fix a subset $\partial_D\Omega$ of $\partial\Omega$, on which we want to prescribe a *Dirichlet boundary condition* for u. We assume that $\partial_D\Omega$ has a *finite number of connected components*.

Given a function g on $\partial_D\Omega$, we consider the boundary condition $u = g$ on $\partial_D\Omega\backslash K$. We can not prescribe a Dirichlet boundary condition on $\partial_D\Omega \cap K$, because the boundary displacement is not transmitted through the crack, if the crack touches the boundary. Assuming that *the fracture is traction free* (and, in particular, without friction), the displacement u in $\Omega\backslash K$ is obtained by *minimizing (1.1) under the boundary condition* $u = g$ *on* $\partial_D\Omega\backslash K$. The *total energy* relative to the boundary displacement g and to the crack determined by K is therefore

$$\mathcal{E}(g,K) = \min_v \Big\{ \int_{\Omega\backslash K} |\nabla v|^2 dx + \mathcal{H}^1(K) : v = g \text{ on } \partial_D\Omega\backslash K \Big\}. \tag{1.3}$$

As K is not assumed to be smooth, we have to be careful in the precise mathematical formulation of this minimum problem, which is given at the beginning of Section 4. The corresponding existence result is based on some properties of the *Deny-Lions spaces*, that are described in Section 2.

In the theory developed in [21] a crack with finite surface energy is any compact subset K of $\overline{\Omega}$ with $\mathcal{H}^1(K) < +\infty$. For technical reasons, that will be explained later, we propose a variant of this model, where we prescribe an a priori bound on the number of connected components of the cracks. Without this restriction, some convergence arguments used in the proof of our existence result are not justified by the present development of the mathematical theories related to this subject.

We now describe our model of *quasi-static irreversible evolution of a fracture* under the action of a *time dependent boundary displacement* $g(t)$, $0 \le t \le 1$. As usual, we assume that $g(t)$ can be extended to a function, still denoted by $g(t)$, which belongs to the Sobolev space $H^1(\Omega)$. In addition, we assume that the function $t \mapsto g(t)$ is *absolutely continuous* from $[0,1]$ into $H^1(\Omega)$. Given an integer $m \ge 1$, let $\mathcal{K}_m^f(\overline{\Omega})$ be the set of all compact subsets K of $\overline{\Omega}$ having at most m connected components and with $\mathcal{H}^1(K) < +\infty$. Following the ideas of [21], given an initial crack $K_0 \in \mathcal{K}_m^f(\overline{\Omega})$, we look for an *increasing family* $K(t)$, $0 \le t \le 1$, of cracks in $\mathcal{K}_m^f(\overline{\Omega})$, such that for any time $t \in (0,1]$ the crack $K(t)$ *minimizes the total energy* $\mathcal{E}(g(t), K)$ among all cracks in $\mathcal{K}_m^f(\overline{\Omega})$ which contain all previous cracks $K(s)$, $s < t$. For $t = 0$ we assume that $K(0)$ minimizes $\mathcal{E}(g(0), K)$ among all cracks in $\mathcal{K}_m^f(\overline{\Omega})$ which contain K_0.

This minimality condition for every time t is inspired by Griffith's analysis of the energy balance. The constraint given by the presence of the previous cracks reflects the *irreversibility of the evolution* and the *absence of a healing process*. In addition to this minimality condition we require also that $\frac{d}{ds}\mathcal{E}(g(t), K(s))|_{s=t} = 0$ for a.e. $t \in [0,1]$. In the special case $g(t) = t\,h$ for a given function $h \in H^1(\Omega)$, we will see (Proposition 7.14) that the last condition implies the third condition considered in Definition 2.9 of [21]: $\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(t), K(s))$ for $s < t$.

In Section 7 we prove the following existence result.

Theorem 1.1. *Let* $g \in AC([0,1]; H^1(\Omega))$ *and let* $K_0 \in \mathcal{K}_m^f(\overline{\Omega})$. *Then there exists a function* $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ *such that*

(a) $$K_0 \subset K(s) \subset K(t) \text{ for } 0 \le s \le t \le 1\,,$$

(b) $$\mathcal{E}(g(0),K(0)) \le \mathcal{E}(g(0),K) \quad \forall\, K \in \mathcal{K}^f_m(\overline{\Omega}),\ \ K \supset K_0\,,$$

(c) $$\textit{for } 0<t\le 1 \quad \mathcal{E}(g(t),K(t)) \le \mathcal{E}(g(t),K) \quad \forall\, K \in \mathcal{K}^f_m(\overline{\Omega}),\ \ K \supset \textstyle\bigcup_{s<t} K(s)\,,$$

(d) $$t \mapsto \mathcal{E}(g(t),K(t)) \textit{ is absolutely continuous on } [0,1]\,,$$

(e) $$\frac{d}{ds}\mathcal{E}(g(t),K(s))\Big|_{s=t} = 0 \quad \textit{for a.e. } t\in[0,1]\,.$$

Moreover every function $K\colon[0,1]\to\mathcal{K}^f_m(\overline{\Omega})$ *which satisfies (a)–(e) satisfies also*

(f) $$\frac{d}{dt}\mathcal{E}(g(t),K(t)) = 2\int_{\Omega\setminus K(t)} \nabla u(t)\,\nabla\dot g(t)\,dx \qquad \textit{for a.e. } t\in[0,1]\,,$$

where $u(t)$ *is a solution of the minimum problem (1.3) which defines* $\mathcal{E}(g(t),K(t))$, *and* $\dot g(t)$ *is the time derivative of the function* $g(t)$.

If $g(0)=0$, we can prove that there exists a solution of problem (a)–(e) with $K(0)=K_0$ (Remark 7.13). We underline that, although we can not exclude that the surface energy $\mathcal{H}^1(K(t))$ may present some jump discontinuities in time (see [21, Section 4.3]), in our result *the total energy is always an absolutely continuous function of time* by condition (d).

If $\partial_D\Omega$ is sufficiently smooth, we can integrate by parts the right hand side of (f) and, taking into account the Euler equation satisfied by $u(t)$, we obtain

$$\frac{d}{dt}\mathcal{E}(g(t),K(t)) = 2\int_{\partial_D\Omega\setminus K(t)} \frac{\partial u(t)}{\partial\nu}\,\dot g(t)\,d\mathcal{H}^1 \qquad \text{for a.e. } t\in[0,1]\,, \tag{1.4}$$

where ν is the outer unit normal to $\partial\Omega$. Since the right hand side of (1.4) is the power of the force exerted on the boundary to obtain the displacement $g(t)$ on $\partial_D\Omega\setminus K(t)$, equality (1.4) expresses the *conservation of energy* in our quasi-static model, where all kinetic effects are neglected.

The proof of this existence result is obtained by a time discretization process. Given a time step $\delta>0$, for every integer $i\ge 0$ we set $t^\delta_i := i\delta$ and $g^\delta_i := g(t^\delta_i)$. We define K^δ_i, inductively, as a solution of the minimum problem

$$\min_K \left\{\mathcal{E}(g^\delta_i,K) : K\in\mathcal{K}^f_m(\overline{\Omega}),\ K\supset K^\delta_{i-1}\right\}, \tag{1.5}$$

where we set $K^\delta_{-1}=K_0$.

Let u^δ_i be a solution of the minimum problem (1.3) which defines $\mathcal{E}(g^\delta_i,K^\delta_i)$. On $[0,1]$ we define the step functions K_δ and u_δ by setting $K_\delta(t) := K^\delta_i$ and $u_\delta(t) := u^\delta_i$ for $t^\delta_i \le t < t^\delta_{i+1}$.

Using a standard monotonicity argument, we prove that there exists a sequence (δ_k) converging to 0 such that, for every $t\in[0,1]$, $K_\delta(t)$ converges to a compact set $K(t)$ in the Hausdorff metric as $\delta\to 0$ along this sequence. Then we can apply the results on the convergence of the solutions to mixed boundary value problems in cracked domains established in Section 5, and we prove that, if $u(t)$ is a solution of the minimum problem (1.3) which defines $\mathcal{E}(g(t),K(t))$, then $\nabla u_\delta(t)$ converges to $\nabla u(t)$ strongly in $L^2(\Omega,\mathbb{R}^2)$ as $\delta\to 0$ along the same sequence considered above.

The technical hypothesis that the sets $K_\delta(t)$ have no more than m connected components plays a crucial role here. Indeed, if this hypothesis is dropped, the convergence in the Hausdorff metric of the cracks $K_\delta(t)$ to the crack $K(t)$ does not imply the convergence of the corresponding solutions of the minimum problems, as shown by many examples in homogenization theory, that can be found, e.g., in [26], [31], [15], [2], [14]. These papers show also that this hypothesis would not be enough in dimension larger than two.

The results of Section 5 are related to those obtained by A. Chambolle and F. Doveri in [12] and by D. Bucur and N. Varchon in [8] and [9], which deal with the case of a pure Neumann boundary condition. Since we impose a Dirichlet boundary condition on $\partial_D\Omega\setminus K_\delta(t)$ and a Neumann boundary condition on the rest of the boundary, our results

can not be deduced easily from these papers, so we give an independent proof, which uses the duality argument of [9].

From this convergence result and from an approximation lemma with respect to the Hausdorff metric, proved in Section 3, we obtain properties (a), (b), (c), (e), and (f) in integrated form, which implies (d).

The time discretization process described above turns out to be a useful tool for the proof of the existence of a solution $K(t)$ of the problem considered in Theorem 1.1, and provides also an effective way for the numerical approximation of this solution (see [6]), since many algorithms have been developed for the numerical solution of minimum problems of the form (1.5) (see, e.g., [3], [32], [33], [4], [11], [5]).

In Section 8 we study the motion of the tips of the time dependent crack $K(t)$ obtained in Theorem 1.1, assuming that, in some open interval $(t_0, t_1) \subset [0,1]$, the crack $K(t)$ has a fixed number of tips, that these tips move smoothly, and that their paths are simple, disjoint, and do not intersect $K(t_0)$. We prove (Theorem 8.4) that in this case *Griffith's criterion for crack growth* is valid in our model: the absolute value of the *stress intensity factor* (see Theorem 8.1 and Remark 8.2) of the solution $u(t)$ is less than or equal to 1 at each tip for every $t \in (t_0, t_1)$, and it is equal to 1 at a given tip for almost every instant $t \in (t_0, t_1)$ in which the tip moves with positive velocity.

2. NOTATION AND PRELIMINARIES

Given an open subset A of $\mathbb{R}^2$, we say that A has a Lipschitz boundary at a point $x \in \partial A$ if A is the sub-graph of a Lipschitz function near x, in the sense that there exist an orthogonal coordinate system (y_1, y_2), a rectangle $U = (a,b) \times (c,d)$ containing x, and a Lipschitz function $\Phi\colon (a,b) \to (c,d)$, such that $A \cap U = \{y \in U : y_2 < \Phi(y_1)\}$. The set of all these points x is the *Lipschitz part of the boundary* and will be denoted by $\partial_L A$. If $\partial_L A = \partial A$, we say that A has a Lipschitz boundary.

Besides the Sobolev space $H^1(A)$ we shall use also the *Deny-Lions space* $L^{1,2}(A) := \{u \in L^2_{loc}(A) \mid \nabla u \in L^2(A; \mathbb{R}^2)\}$, which coincides with the space of all distributions u on A such that $\nabla u \in L^2(A;\mathbb{R}^2)$ (see, e.g., [27, Theorem 1.1.2]). For the proof of the following result we refer, e.g., to [27, Section 1.1.13].

Proposition 2.1. *The set* $\{\nabla u : u \in L^{1,2}(A)\}$ *is closed in* $L^2(A;\mathbb{R}^2)$.

Under some regularity assumptions on the boundary, the following result holds.

Proposition 2.2. *Let* $u \in L^{1,2}(A)$ *and* $x \in \partial_L A$. *Then there exists a neighbourhood* U *of* x *such that* $u|_{A\cap U} \in H^1(A \cap U)$. *In particular, if* A *is bounded and has a Lipschitz boundary, then* $L^{1,2}(A) = H^1(A)$.

Proof. Let U be the rectangle given by the definition of Lipschitz boundary. It is easy to check that $A \cap U$ has a Lipschitz boundary. The conclusion follows now from the Corollary to Lemma 1.1.11 in [27]. ☐

We recall some properties of the functions in the spaces $H^1(A)$ and $L^{1,2}(A)$, which are related to the notion of capacity. For more details we refer to [18], [25], [27], and [38].

Definition 2.3. *Let* B *be a bounded open set in* $\mathbb{R}^2$. *The capacity of an arbitrary subset* E *of* B *is defined as*

$$\mathrm{cap}(E,B) := \inf_{u \in \mathcal{U}_E^B} \int_B |\nabla u|^2 \, dx \,,$$

where $\mathcal{U}_E^B$ *is the set of all functions* $u \in H^1_0(B)$ *such that* $u \geq 1$ *a.e. in a neighbourhood of* E.

We say that a property is true *quasi-everywhere* on a set $E \subset B$, and write *q.e.*, if it holds on E except on a set of capacity zero. As usual, the expression almost everywhere,

abbreviated as a.e., refers to the Lebesgue measure. A function $u\colon E \to \overline{\mathbb{R}}$ is said to be *quasi-continuous* on E if for every $\varepsilon > 0$ there exists an open set U_ε, with $\mathrm{cap}(U_\varepsilon, B) < \varepsilon$, such that $u|_{E\setminus U_\varepsilon}$ is continuous on $E\setminus U_\varepsilon$. It is easy to prove that both notions of quasi-everywhere and quasi-continuity do not depend on B.

It is known that every function $u \in L^{1,2}(A)$ has a *quasi-continuous representative* $\tilde{u}$, which is uniquely defined q.e. on $A \cup \partial_L A$, and satisfies

$$\lim_{\rho\to 0} \fint_{B_\rho(x)\cap A} |u(y) - \tilde{u}(x)|\, dy = 0 \qquad \text{for q.e. } x \in A \cup \partial_L A\,, \tag{2.1}$$

where $\fint$ denotes the average and $B_\rho(x)$ is the open ball with centre x and radius ρ. If $u_n \to u$ strongly in $H^1(A)$, then a subsequence of $(\tilde{u}_n)$ converges to $\tilde{u}$ q.e. in $A \cup \partial_L A$. If u, $v \in L^{1,2}(A)$ and their traces coincide $\mathcal{H}^1$-a.e. on $\partial_L A$, then $\tilde{u}$ and $\tilde{v}$ coincide q.e. on $\partial_L A$.

In the quoted books the quasi-continuous representatives are defined only on A. The straightforward definition of $\tilde{u}$ on $\partial_L A$ relies on the existence of extension operators for Lipschitz domains; the q.e. uniqueness of $\tilde{u}$ on $\partial_L A$ can be deduced from (2.1). To simplify the notation we shall always identify each function $u \in L^{1,2}(A)$ with its quasi-continuous representative $\tilde{u}$.

Propositions 2.1 and 2.2 imply the following result.

Corollary 2.4. *Assume that A is connected, and let Γ be a non-empty relatively open subset of ∂A with $\Gamma \subset \partial_L A$. Then the space $L^{1,2}_0(A,\Gamma) := \{u \in L^{1,2}(A) : u = 0 \ \text{ q.e. on } \Gamma\}$ is a Hilbert space with the norm $\|\nabla u\|_{L^2(A;\mathbb{R}^2)}$. Moreover, if (u_n) is a bounded sequence in $L^{1,2}_0(A,\Gamma)$, then there exist a subsequence, still denoted by (u_n), and a function $u \in L^{1,2}_0(A,\Gamma)$ such that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(A;\mathbb{R}^2)$.*

Proof. Let (v_n) be a Cauchy sequence in $L^{1,2}_0(A,\Gamma)$. We can construct an increasing sequence (A_k) of connected open sets with Lipschitz boundary such that $A = \bigcup_k A_k$, and $\Gamma = \bigcup_k (\partial A_k \cap \partial A)$.

By Proposition 2.2 the functions v_n belong to $H^1(A_k)$ and $v_n = 0$ q.e. on $\partial A_k \cap \partial A$. As $\mathcal{H}^1(\partial A_k \cap \partial A) > 0$ for k large enough, by the Poincaré inequality (v_n) is a Cauchy sequence in $H^1(A_k)$, and therefore it converges strongly in $H^1(A_k)$ to a function v with $v = 0$ q.e. on $\partial A_k \cap \partial A$. It is then easy to construct a function $v \in L^{1,2}(A)$ such that $v = 0$ q.e. on Γ and $v_n \to v$ strongly in $H^1(A_k)$ for every k. As (∇v_n) converges strongly in $L^2(A;\mathbb{R}^2)$, we conclude that $v_n \to v$ strongly in $L^{1,2}_0(A;\Gamma)$.

Let (u_n) be a bounded sequence in $L^{1,2}_0(A,\Gamma)$. As in the previous part of the proof we deduce that (u_n) is bounded in $H^1(A_k)$ for every k. By a diagonal argument we can prove that there exist a subsequence, still denoted by (u_n), and a function $u \in L^{1,2}(A)$ such that $u_n \rightharpoonup u$ weakly in $H^1(A_k)$ for every k. Then a sequence of convex combinations of the functions u_n converges to u strongly in $H^1(A_k)$. This implies $u = 0$ q.e. on $\partial A_k \cap \partial A$ for every k, hence $u \in L^{1,2}_0(A,\Gamma)$. As (∇u_n) is bounded in $L^2(A;\mathbb{R}^2)$, we conclude that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(A;\mathbb{R}^2)$. $\square$

Proposition 2.5. *Let $u \in L^{1,2}(A)$ and let C_1 and C_2 be two connected subsets of $A \cup \partial_L A$ with $\overline{C}_1 \cap \overline{C}_2 \neq \emptyset$. Assume that u is constant q.e. on C_i for $i = 1, 2$. Then u is constant q.e. on $C_1 \cup C_2$.*

Proof. We may assume that C_1 and C_2 have more than one point, since otherwise the statement is trivial. Let us denote the constant values of u on C_1 and C_2 by c_1 and c_2 respectively, and let us fix $x \in \overline{C}_1 \cap \overline{C}_2$. Since $x \in A \cup \partial_L A$, we may assume that u belongs to $H^1(B_r(x))$ for some $r > 0$ (we use an extension operator if $x \in \partial_L A$), and that $C_i \cap \partial B_\rho(x) \neq \emptyset$ for $i = 1, 2$ and $0 < \rho < r$. Hence for almost every $\rho \in (0, r)$ (the quasi-continuous representative of) u takes the values c_1 and c_2 in two distinct points of

$\partial B_\rho(x)$. This implies

$$\int_{\partial B_\rho(x)} |\nabla u|^2 \, d\mathcal{H}^1 \geq \frac{(c_2 - c_1)^2}{\pi \rho},$$

which yields $\nabla u \notin L^2(B_r(x); \mathbb{R}^2)$, in contradiction with our assumption. □

We conclude this section by stating a property of connected sets with finite length.

Proposition 2.6. *Let C be a connected subset of $\mathbb{R}^2$. Then $\mathcal{H}^1(\overline{C}) = \mathcal{H}^1(C)$.*

Proof. It is clearly enough to prove the statement when $\mathcal{H}^1(C) < +\infty$. The following concise argument was suggested by Luigi Ambrosio. If x, $y \in C$, then $\mathcal{H}^1(C) \geq |x-y|$ (the classical proof, see e.g., [19, Lemma 3.4], does not need the hypothesis that C is compact). Therefore $\mathcal{H}^1(C \cap B_\rho(x)) \geq \rho$ for every $x \in \overline{C}$ and $0 < \rho < \mathrm{diam}(C)/2$. This implies that $\mathcal{H}^1(\overline{C} \setminus C) = 0$ by a standard argument based on the Besicovitch covering lemma (see [20, 2.10.19(4)]). □

3. Hausdorff measure and Hausdorff convergence

Throughout the paper Ω is a fixed *bounded connected open* subset of $\mathbb{R}^2$ with *Lipschitz boundary*. In this section we study the behaviour of the Hausdorff measure $\mathcal{H}^1$ along suitable sequences of compact sets which converge in the Hausdorff metric.

Let $\mathcal{K}(\overline{\Omega})$ be the set of all compact subsets of $\overline{\Omega}$, and let $\mathcal{K}^f(\overline{\Omega}) := \{K \in \mathcal{K}(\overline{\Omega}) : \mathcal{H}^1(K) < +\infty\}$. Given an integer $m \geq 1$, let $\mathcal{K}_m(\overline{\Omega})$ be the set of all compact subsets of $\overline{\Omega}$ with at most m connected components, and let $\mathcal{K}_m^f(\overline{\Omega}) := \{K \in \mathcal{K}_m(\overline{\Omega}) : \mathcal{H}^1(K) < +\infty\}$. For every $\lambda \geq 0$ we consider also the set $\mathcal{K}_m^\lambda(\overline{\Omega}) := \{K \in \mathcal{K}_m(\overline{\Omega}) : \mathcal{H}^1(K) \leq \lambda\}$.

We recall that the *Hausdorff distance* between K_1, $K_2 \in \mathcal{K}(\overline{\Omega})$ is defined by

$$d_H(K_1, K_2) := \max\big\{ \sup_{x \in K_1} \mathrm{dist}(x, K_2), \sup_{y \in K_2} \mathrm{dist}(y, K_1) \big\},$$

with the conventions $\mathrm{dist}(x, \emptyset) = \mathrm{diam}(\Omega)$ and $\sup \emptyset = 0$, so that $d_H(\emptyset, K) = 0$ if $K = \emptyset$ and $d_H(\emptyset, K) = \mathrm{diam}(\Omega)$ if $K \neq \emptyset$. We say that $K_n \to K$ in the Hausdorff metric if $d_H(K_n, K) \to 0$. The following compactness theorem is well-known (see, e.g., [34, Blaschke's Selection Theorem]).

Theorem 3.1. *Let (K_n) be a sequence in $\mathcal{K}(\overline{\Omega})$. Then there exists a subsequence which converges in the Hausdorff metric to a set $K \in \mathcal{K}(\overline{\Omega})$.*

It is well-known that, in general, the Hausdorff measure is not lower semicontinuous on $\mathcal{K}(\overline{\Omega})$ with respect to the convergence in the Hausdorff metric. When all sets are connected, we have the following lower semicontinuity theorem, whose proof can be obtained as in Theorem 10.19 of [28].

Theorem 3.2 (Gołąb's Theorem)**.** *Let (K_n) be a sequence in $\mathcal{K}_1(\overline{\Omega})$ which converges to K in the Hausdorff metric. Then $K \in \mathcal{K}_1(\overline{\Omega})$ and*

$$\mathcal{H}^1(K \cap U) \leq \liminf_{n \to \infty} \mathcal{H}^1(K_n \cap U)$$

for every open set $U \subset \mathbb{R}^2$.

Gołąb's Theorem says that, for every $\lambda < +\infty$, $\mathcal{K}_1^\lambda(\overline{\Omega})$ is closed under convergence in the Hausdorff metric. In the next corollary we extend this result to $\mathcal{K}_m^\lambda(\overline{\Omega})$.

Corollary 3.3. *Let $m \geq 1$ and let (K_n) be a sequence in $\mathcal{K}_m(\overline{\Omega})$ which converges to K in the Hausdorff metric. Then $K \in \mathcal{K}_m(\overline{\Omega})$ and*

$$\mathcal{H}^1(K \cap U) \leq \liminf_{n \to \infty} \mathcal{H}^1(K_n \cap U)$$

for every open set $U \subset \mathbb{R}^2$.

Proof. Let $K_n^1,\dots,K_n^{k_n}$ be the connected components of K_n. As $k_n\le m$, there exists $k\le m$ such that, up to a subsequence, $k_n=k$ for all n. By Theorem 3.1 we may also assume that $K_n^1\to\widehat K^1,\ \dots,\ K_n^k\to\widehat K^k$ in the Hausdorff metric, where $\widehat K^1,\dots,\widehat K^k$ are compact and connected.

We claim that

$$K\subset\widehat K^1\cup\dots\cup\widehat K^k\,. \tag{3.1}$$

Indeed, for every $x\in K$ there exists a sequence $x_n\to x$ such that $x_n\in K_n$, which implies $x_n\in K_n^{i_n}$ for some i_n between 1 and k. Hence there exists i such that $i_n=i$ for infinitely many indices n, and, consequently, $x\in\widehat K^i$. This proves (3.1), which implies that K has at most $k\le m$ connected components.

By Gołąb's Theorem 3.2 we have

$$\mathcal H^1(\widehat K^j\cap U)\le\liminf_{n\to\infty}\mathcal H^1(K_n^j\cap U)$$

for $j=1,\dots,k$. The conclusion follows now from (3.1). □

We shall use also the following consequence of Corollary 3.3.

Corollary 3.4. *Let (H_n) be a sequence in $\mathcal K(\overline\Omega)$ which converges to H in the Hausdorff metric. Let $m\ge1$ and let (K_n) be a sequence in $\mathcal K_m(\overline\Omega)$ which converges to K in the Hausdorff metric. Then*

$$\mathcal H^1(K\backslash H)\le\liminf_{n\to\infty}\mathcal H^1(K_n\backslash H_n)\,. \tag{3.2}$$

Proof. Given $\varepsilon>0$, let $H^\varepsilon:=\{x\in\overline\Omega:\mathrm{dist}(x,H)\le\varepsilon\}$. As $H_n\subset H^\varepsilon$ for n large enough, we have $K_n\backslash H^\varepsilon\subset K_n\backslash H_n$. Applying Corollary 3.3 with $U=\mathbb R^2\backslash H^\varepsilon$ we get

$$\mathcal H^1(K\backslash H^\varepsilon)\le\liminf_{n\to\infty}\mathcal H^1(K_n\backslash H^\varepsilon)\le\liminf_{n\to\infty}\mathcal H^1(K_n\backslash H_n)\,.$$

Passing to the limit as $\varepsilon\to0$ we obtain (3.2). □

In Section 7 we shall use the following approximation result.

Lemma 3.5. *Let p and m be positive integers, let (H_n) be a sequence in $\mathcal K_p^f(\overline\Omega)$ which converges in the Hausdorff metric to $H\in\mathcal K_p^f(\overline\Omega)$, and let K be an element of $\mathcal K_m^f(\overline\Omega)$ with $K\supset H$. Then there exists a sequence (K_n) in $\mathcal K_m^f(\overline\Omega)$ such that $K_n\to K$ in the Hausdorff metric, $H_n\subset K_n$, and $\mathcal H^1(K_n\backslash H_n)\to\mathcal H^1(K\backslash H)$.*

To prove Lemma 3.5 we need the following three lemmas.

Lemma 3.6. *Let $H\in\mathcal K_1(\overline\Omega)$ and let (H_n) be a sequence in $\mathcal K_p(\overline\Omega)$ which converges to H in the Hausdorff metric. Then there exists a sequence $(\widehat H_n)$ in $\mathcal K_1(\overline\Omega)$ such that $\widehat H_n\to H$ in the Hausdorff metric, $H_n\subset\widehat H_n$, and $\mathcal H^1(\widehat H_n\backslash H_n)\to0$.*

Proof. Passing to a subsequence, we may assume, as in the first part of the proof of Corollary 3.3, that there exists a constant $k\le p$ such that every H_n has exactly k connected components $H_n^1,\dots,H_n^k$ and $H_n^1\to\widehat H^1,\ \dots,\ H_n^k\to\widehat H^k$ in the Hausdorff metric, where $\widehat H^1,\dots,\widehat H^k$ are compact and connected and $H=\widehat H^1\cup\dots\cup\widehat H^k$.

As H is connected, there exists a finite family of indices $(\sigma_j)_{0\le j\le\ell}$, with $\{\sigma_0,\dots,\sigma_\ell\}=\{1,\dots,k\}$, such that $\widehat H^{\sigma_{j-1}}\cap\widehat H^{\sigma_j}\ne\emptyset$ for $j=1,\dots,\ell$. Let us fix a point $x^j\in\widehat H^{\sigma_{j-1}}\cap\widehat H^{\sigma_j}$. By the convergence in the Hausdorff metric there exist $x_n^j\in H_n^{\sigma_{j-1}}$ and $y_n^j\in H_n^{\sigma_j}$ such that $x_n^j\to x^j$ and $y_n^j\to x^j$ as $n\to\infty$.

Since Ω has a Lipschitz boundary, there exist arcs X_n^j and Y_n^j in $\overline\Omega$, connecting x_n^j to x^j and y_n^j to x^j respectively, such that $\mathcal H^1(X_n^j)\to0$ and $\mathcal H^1(Y_n^j)\to0$ as $n\to\infty$. Let us define

$$\widehat H_n:=H_n\cup\bigcup_{j=1}^{\ell}X_n^j\cup\bigcup_{j=1}^{\ell}Y_n^j\,.$$

It is clear that $\widehat{H}_n \to H$ in the Hausdorff metric and that $\mathcal{H}^1(\widehat{H}_n \setminus H_n) \to 0$. Since

$$\widehat{H}_n = H_n^{\sigma_0} \cup X_n^1 \cup Y_n^1 \cup H_n^{\sigma_1} \cup \cdots \cup H_n^{\sigma_{\ell-1}} \cup X_n^\ell \cup Y_n^\ell \cup H_n^{\sigma_\ell},$$

we conclude that $\widehat{H}_n$ is connected. □

Lemma 3.7. *Let $K \in \mathcal{K}_1^f(\overline{\Omega})$ and let H be a non-empty compact subset of K with $p \geq 2$ connected components $H^1, \dots, H^p$. Then there exist a finite family of indices $(\sigma_j)_{0 \leq j \leq \ell}$, with $\{\sigma_0, \dots, \sigma_\ell\} = \{1, \dots, p\}$, and a family $(\Gamma_j)_{1 \leq j \leq \ell}$ of connected components of $K \setminus H$, such that $\overline{\Gamma}_j$ connects $H^{\sigma_{j-1}}$ with H^{σ_j} for $j = 1, \dots, \ell$.*

Proof. It is clear that $K \setminus H \neq \emptyset$, since otherwise H has exactly one connected component. Since K is locally connected (see, e.g., [12, Lemma 1]), and $K \setminus H$ is open in K, the connected components C of $K \setminus H$ are open in K. Since each C is closed in $K \setminus H$, we have $C = \overline{C} \cap (K \setminus H)$. If $C = \overline{C}$, then K would contain an open, closed, and non-empty proper subset (recall that $H \neq \emptyset$), which contradicts the fact that K is connected. Therefore $C \neq \overline{C}$. As $\overline{C} \cap (K \setminus H) = C$, we conclude that $\emptyset \neq \overline{C} \setminus C \subset H$. Therefore $\overline{C} \cap H \neq \emptyset$ for every connected component C of $K \setminus H$.

For $j = 1, \dots, p$ let $\widehat{K}^j$ be the union of H^j and of all the connected components C of $K \setminus H$ such that $\overline{C} \cap H^j \neq \emptyset$. To prove that $\widehat{K}^j$ is open in K, we fix a sequence (x_n) in $K \setminus \widehat{K}^j$ which converges to a point $x \in K$. If $x_n \in H \setminus H^j$ for infinitely many indices n, then $x \in H \setminus H^j$, hence $x \notin \widehat{K}^j$. If there exists a connected component C_0 of $K \setminus H$ such that $\overline{C}_0 \cap H^j = \emptyset$ and $x_n \in \overline{C}_0$ for infinitely many indices n, then $x \in \overline{C}_0$; this implies $x \notin \widehat{K}^j$, since $\overline{C}_0 \cap C = \emptyset$ for every connected component C of $K \setminus H$ with $\overline{C} \cap H^j \neq \emptyset$. In the other cases there exists a sequence (C_n) of pairwise disjoint connected components of $K \setminus H$, with $\overline{C}_n \cap H^j = \emptyset$, such that, up to a subsequence, $x_n \in \overline{C}_n$. As $\mathcal{H}^1(K) < +\infty$, by Proposition 2.6 $\mathcal{H}^1(\overline{C}_n) = \mathcal{H}^1(C_n) \to 0$, hence $\operatorname{dist}(x_n, H \setminus H^j) \to 0$, which gives $x \in H \setminus H^j$, so that $x \notin \widehat{K}^j$ also in this case. Therefore $\widehat{K}^j$ is open in K.

Since $\overline{C} \cap H \neq \emptyset$ for every connected component C of $K \setminus H$, we have $K = \widehat{K}^1 \cup \cdots \cup \widehat{K}^p$. As K is connected, there exists a finite family of indices $(\sigma_j)_{0 \leq j \leq \ell}$, with $\{\sigma_0, \dots, \sigma_\ell\} = \{1, \dots, p\}$, such that $\sigma_{j-1} \neq \sigma_j$ and $\widehat{K}^{\sigma_{j-1}} \cap \widehat{K}^{\sigma_j} \neq \emptyset$ for $j = 1, \dots, \ell$. As $H^{\sigma_{j-1}} \cap H^{\sigma_j} = \emptyset$, there exists a connected component Γ_j of $K \setminus H$ such that $\Gamma_j \subset \widehat{K}^{\sigma_{j-1}} \cap \widehat{K}^{\sigma_j}$ and, consequently, $H^{\sigma_{j-1}} \cap \overline{\Gamma}_j \neq \emptyset \neq H^{\sigma_j} \cap \overline{\Gamma}_j$. □

Lemma 3.8. *Let p be a positive integer, let (H_n) be a sequence in $\mathcal{K}_p^f(\overline{\Omega})$ which converges in the Hausdorff metric to $H \in \mathcal{K}_p^f(\overline{\Omega})$, and let K be an element of $\mathcal{K}_1^f(\overline{\Omega})$ with $K \supset H$. Then there exists a sequence (K_n) in $\mathcal{K}_1^f(\overline{\Omega})$ such that $K_n \to K$ in the Hausdorff metric, $H_n \subset K_n$, and $\mathcal{H}^1(K_n \setminus H_n) \to \mathcal{H}^1(K \setminus H)$.*

Proof. If $H = \emptyset$, we just define $K_n := K$ and notice that $H_n = \emptyset$ for n large enough.

Assume now $H \neq \emptyset$ and let $H^1, \dots, H^k$, $k \leq p$, be its connected components. If $k = 1$ we set $\widehat{K} := H = H^1$. If $k \geq 2$, by Lemma 3.7 there exist a finite family of indices $(\sigma_j)_{0 \leq j \leq \ell}$, with $\{\sigma_0, \dots, \sigma_\ell\} = \{1, \dots, k\}$, and a family $(\Gamma_j)_{1 \leq j \leq \ell}$ of connected components of $K \setminus H$, such that $H^{\sigma_{j-1}} \cap \overline{\Gamma}_j \neq \emptyset \neq H^{\sigma_j} \cap \overline{\Gamma}_j$ for $j = 1, \dots, \ell$; in this case we set

$$\widehat{K} := H \cup \bigcup_{j=1}^{\ell} \overline{\Gamma}_j .$$

In both cases we want to construct a sequence $(\widehat{K}_n)$ in $\mathcal{K}_1^f(\overline{\Omega})$ which converges to $\widehat{K}$ in the Hausdorff metric and such that $H_n \subset \widehat{K}_n$ and

$$\limsup_{n \to \infty} \mathcal{H}^1(\widehat{K}_n \setminus H_n) \leq \mathcal{H}^1(\widehat{K} \setminus H) . \tag{3.3}$$

Let us fix $\varepsilon > 0$ such that the sets $\{x \in \overline{\Omega} : \mathrm{dist}(x, H^i) \le \varepsilon\}$, $i = 1, \dots, k$, are pairwise disjoint, and let

$$\widetilde{H}^i_n := \{x \in H_n : \mathrm{dist}(x, H^i) \le \varepsilon\}\,.$$

It is easy to see that $\widetilde{H}^i_n \in \mathcal{K}^f_p(\overline{\Omega})$ and $H_n = \widetilde{H}^1_n \cup \cdots \cup \widetilde{H}^k_n$ for n large enough, and that $(\widetilde{H}^i_n)$ converges to H^i in the Hausdorff metric as $n \to \infty$. By Lemma 3.6 there exists a sequence $(\widehat{H}^i_n)$ in $\mathcal{K}^f_1(\overline{\Omega})$ such that $\widehat{H}^i_n \to H^i$ in the Hausdorff metric, $\widetilde{H}^i_n \subset \widehat{H}^i_n$, and $\mathcal{H}^1(\widehat{H}^i_n \setminus \widetilde{H}^i_n) \to 0$.

If $k = 1$ we define $\widehat{K}_n := \widehat{H}^1_n$.

If $k \ge 2$, for every $j = 1, \dots, \ell$ we fix two points $x^j \in H^{\sigma_{j-1}} \cap \overline{\Gamma}_j$ and $y^j \in H^{\sigma_j} \cap \overline{\Gamma}_j$. By the convergence in the Hausdorff metric there exist $x^j_n \in \widehat{H}^{\sigma_{j-1}}_n$ and $y^j_n \in \widehat{H}^{\sigma_j}_n$ such that $x^j_n \to x^j$ and $y^j_n \to y^j$ as $n \to \infty$. Since Ω has a Lipschitz boundary, there exist arcs X^j_n and Y^j_n in $\overline{\Omega}$, connecting x^j_n to x^j and y^j_n to y^j respectively, such that $\mathcal{H}^1(X^j_n) \to 0$ and $\mathcal{H}^1(Y^j_n) \to 0$ as $n \to \infty$. Let us define

$$\widehat{K}_n := \bigcup_{i=1}^{k} \widehat{H}^i_n \cup \bigcup_{j=1}^{\ell} X^j_n \cup \bigcup_{j=1}^{\ell} \overline{\Gamma}_j \cup \bigcup_{j=1}^{\ell} Y^j_n\,.$$

In both cases $k = 1$ and $k \ge 2$ it is clear that $\widehat{K}_n \to \widehat{K}$ in the Hausdorff metric and that (3.3) holds, since by Proposition 2.6 $\mathcal{H}^1(\overline{\Gamma}_j) = \mathcal{H}^1(\Gamma_j)$. As $\widehat{K}_n = \widehat{H}^1_n$ for $k = 1$, and

$$\widehat{K}_n = \widehat{H}^{\sigma_0}_n \cup X^1_n \cup \overline{\Gamma}_1 \cup Y^1_n \cup \widehat{H}^{\sigma_1}_n \cup \cdots \cup \widehat{H}^{\sigma_{\ell-1}}_n \cup X^\ell_n \cup \overline{\Gamma}_\ell \cup Y^\ell_n \cup \widehat{H}^{\sigma_\ell}_n$$

for $k \ge 2$, we conclude that $\widehat{K}_n$ is connected in both cases.

As the connected components C of $K \setminus \widehat{K}$ are connected components of $K \setminus H$, the argument given at the beginning of the proof of Lemma 3.7 shows that each C is open in K and satisfies $\overline{C} \cap H \ne \emptyset$. Since K is separable, the connected components of $K \setminus \widehat{K}$ form a finite or countable sequence (C_i).

For every i we fix a point $z^i \in \overline{C}_i \cap H$. As $H_n \to H$ in the Hausdorff metric, there exists $z^i_n \in H_n$ such that $z^i_n \to z^i$ as $n \to \infty$. Since Ω has a Lipschitz boundary, for every i there exists an arc Z^i_n in $\overline{\Omega}$, connecting z^i_n to z^i, such that $\mathcal{H}^1(Z^i_n) \to 0$ as $n \to \infty$.

If there are infinitely many connected components C_i, there exists a sequence of integers (h_n) tending to ∞ such that

$$\lim_{n\to\infty} \sum_{i=1}^{h_n} \mathcal{H}^1(Z^i_n) = 0\,. \tag{3.4}$$

If there are $h < +\infty$ connected components C_i, (3.4) is true with $h_n = h$ for every n. Let

$$K_n := \widehat{K}_n \cup \bigcup_{i=1}^{h_n} Z^i_n \cup \bigcup_{i=1}^{h_n} \overline{C}_i\,.$$

Then the sets K_n are connected, contain H_n, and converge to K in the Hausdorff metric. As $\mathcal{H}^1(C_i) = \mathcal{H}^1(\overline{C}_i)$ by Proposition 2.6, we have

$$\mathcal{H}^1(K_n \setminus H_n) \le \mathcal{H}^1(\widehat{K}_n \setminus H_n) + \sum_{i=1}^{h_n} \mathcal{H}^1(Z^i_n) + \sum_{i=1}^{h_n} \mathcal{H}^1(C_i)\,,$$

which, together with (3.3) and (3.4), yields

$$\limsup_{n\to\infty} \mathcal{H}^1(K_n \setminus H_n) \le \mathcal{H}^1(\widehat{K} \setminus H) + \mathcal{H}^1(\textstyle\bigcup_i C_i) = \mathcal{H}^1(K \setminus H)\,. \tag{3.5}$$

The opposite inequality for the lower limit follows from Corollary 3.4. $\square$

Proof of Lemma 3.5. Let $K^1, \dots, K^k$, $k \le m$, be the connected components of K. Let us fix $\varepsilon > 0$ such that the sets $\{x \in \overline{\Omega} : \mathrm{dist}(x, K^i) \le \varepsilon\}$, $i = 1, \dots, k$, are pairwise disjoint, and let

$$\widehat{H}^i_n := \{x \in H_n : \mathrm{dist}(x, K^i) \le \varepsilon\}\,.$$

It is easy to see that $\widehat{H}^i_n \in \mathcal{K}^f_p(\overline{\Omega})$ and $H_n = \widehat{H}^1_n \cup \dots \cup \widehat{H}^k_n$ for n large enough, and that $(\widehat{H}^i_n)$ converges to $H^i := H \cap K^i$ in the Hausdorff metric as $n \to \infty$. By Lemma 3.8 there exists a sequence (K^i_n) in $\mathcal{K}^f_1(\overline{\Omega})$ such that $K^i_n \to K^i$ in the Hausdorff metric, $\widehat{H}^i_n \subset K^i_n$, and $\mathcal{H}^1(K^i_n \setminus \widehat{H}^i_n) \to \mathcal{H}^1(K^i \setminus H^i)$. It suffices now to take $K_n := K^1_n \cup \dots \cup K^k_n$. □

4. PROPERTIES OF THE HARMONIC CONJUGATE

In the rest of the paper $\partial_N \Omega$ is a fixed (possibly empty) relatively open subset of $\partial\Omega$, with a finite number of connected components, on which we impose a Neumann boundary condition. Let $\partial_D \Omega := \partial\Omega \setminus \overline{\partial_N \Omega}$, which turns out to be a relatively open subset of $\partial\Omega$, with a finite number of connected components. On this set we want to impose a Dirichlet boundary condition.

Given $K \in \mathcal{K}(\overline{\Omega})$, we consider the following boundary value problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus K\,, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial(\Omega \setminus K) \cap (K \cup \partial_N \Omega)\,. \end{cases} \tag{4.1}$$

By a solution of (4.1) we mean a function u which satisfies the following conditions:

$$\begin{cases} u \in L^{1,2}(\Omega \setminus K)\,, \\ \displaystyle\int_{\Omega \setminus K} \nabla u \, \nabla z \, dx = 0 \quad \forall z \in L^{1,2}(\Omega \setminus K)\,,\ z = 0 \quad \text{q.e. on } \partial_D \Omega \setminus K\,. \end{cases} \tag{4.2}$$

Since no boundary condition is prescribed on $\partial_D \Omega \setminus K$, we do not expect a unique solution to problem (4.1). Given $g \in L^{1,2}(\Omega \setminus K)$, we can prescribe the Dirichlet boundary condition

$$u = g \quad \text{q.e. on } \partial_D \Omega \setminus K\,. \tag{4.3}$$

It is clear that problem (4.2) with the boundary condition (4.3) can be solved separately in each connected component of $\Omega \setminus K$. By Corollary 2.4 and by the Lax-Milgram lemma there exists a unique solution in those components whose boundary meets $\partial_D \Omega \setminus K$, while on the other components the solution is given by an arbitrary constant. Thus the solution is not unique, if there is a connected component whose boundary does not meet $\partial_D \Omega \setminus K$. Note, however, that ∇u is always unique. Moreover, the map $g \mapsto \nabla u$ is linear from $L^{1,2}(\Omega \setminus K)$ into $L^2(\Omega \setminus K; \mathbb{R}^2)$ and satisfies the estimate

$$\int_{\Omega \setminus K} |\nabla u|^2 \, dx \le \int_{\Omega \setminus K} |\nabla g|^2 \, dx\,.$$

By standard arguments on the minimization of quadratic forms it is easy to see that u is a solution of problem (4.2) and satisfies the boundary condition (4.3) if and only if u is a solution of the minimum problem

$$\min_{v \in \mathcal{V}(g,K)} \int_{\Omega \setminus K} |\nabla v|^2 \, dx\,, \tag{4.4}$$

where

$$\mathcal{V}(g, K) := \{v \in L^{1,2}(\Omega \setminus K) : v = g \quad \text{q.e. on } \partial_D \Omega \setminus K\}\,. \tag{4.5}$$

Throughout the paper, given a function $u \in L^{1,2}(\Omega \setminus K)$ for some $K \in \mathcal{K}(\overline{\Omega})$, we always extend ∇u to Ω by setting $\nabla u = 0$ a.e. on K. Note that, however, ∇u is the distributional gradient of u only in $\Omega \setminus K$, and, in general, it does not coincide in Ω with the gradient of an extension of u.

To study the continuous dependence on K of the solutions of problem (4.2) with boundary condition (4.3), we shall use the following lemma.

Lemma 4.1. *Let (K_n) be a sequence in $\mathcal{K}(\overline{\Omega})$ which converges to K in the Hausdorff metric. Let $u_n \in L^{1,2}(\Omega \setminus K_n)$ be a sequence such that $u_n = 0$ q.e. on $\partial_D\Omega \setminus K_n$ and (∇u_n) is bounded in $L^2(\Omega;\mathbb{R}^2)$. Then there exist a subsequence, still denoted by (u_n), and a function $u \in L^{1,2}(\Omega \setminus K)$, such that $u = 0$ q.e. on $\partial_D\Omega \setminus K$ and $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(U;\mathbb{R}^2)$ for every open set $U \subset\subset \Omega \setminus K$. If, in addition, $\mathrm{meas}(K_n) \to \mathrm{meas}(K)$, then $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega;\mathbb{R}^2)$.*

Proof. Let C be a connected component of $\Omega \setminus K$ and let $x \in C$. Given $0 < \varepsilon < \mathrm{dist}(x, \partial C)$, let $N^\varepsilon := \{x \in \mathbb{R}^2 : \mathrm{dist}(x, \partial_N\Omega \cup K) \le \varepsilon\}$ and let C^ε be the connected component of $C \setminus N^\varepsilon$ containing x. For n large enough we have $K_n \subset N^\varepsilon$.

If the boundary of C meets $\partial_D\Omega \setminus K$, let Γ^ε be the relative interior of $\partial C^\varepsilon \cap \partial_D\Omega$ in ∂C^ε. Since ∂C meets $\partial_D\Omega \setminus K$, for ε small enough we have $\Gamma^\varepsilon \neq \emptyset$. As $u_n = 0$ q.e. on Γ^ε, we apply Corollary 2.4 and deduce that there exists a function $u \in L^{1,2}(C^\varepsilon)$, with $u = 0$ q.e. on Γ^ε, such that, up to a subsequence, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(C^\varepsilon;\mathbb{R}^2)$. Since $\varepsilon > 0$ is arbitrary and $C = \bigcup_\varepsilon C_\varepsilon$, we can construct $u \in L^{1,2}(C)$, with $u = 0$ q.e. on $(\partial C \cap \partial_D\Omega) \setminus K$, such that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(U;\mathbb{R}^2)$ for every open set $U \subset\subset C$.

If the boundary of C does not meet $\partial_D\Omega \setminus K$, passing to a subsequence, we can still assume that (∇u_n) converges weakly in $L^2(C^\varepsilon;\mathbb{R}^2)$ to some function $\varphi \in L^2(C^\varepsilon;\mathbb{R}^2)$. Since the space $\{\nabla v : v \in L^{1,2}(C^\varepsilon)\}$ is closed in $L^2(C^\varepsilon;\mathbb{R}^2)$, we conclude that there exists $u \in L^{1,2}(C^\varepsilon)$ such that $\nabla u = \varphi$ a.e. in C^ε, and, as in the previous case, we can construct $u \in L^{1,2}(C)$ such that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(U;\mathbb{R}^2)$ for every open set $U \subset\subset C$.

Therefore we have constructed $u \in L^{1,2}(\Omega \setminus K)$, with $u = 0$ q.e. on $\partial_D\Omega \setminus K$, such that $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(U;\mathbb{R}^2)$ for every open set $U \subset\subset \Omega \setminus K$.

Assume now that $\mathrm{meas}(K_n) \to \mathrm{meas}(K)$ and let $\psi \in L^2(\Omega;\mathbb{R}^2)$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_A |\psi|^2\,dx < \varepsilon^2$ for $\mathrm{meas}(A) < \delta$. Let $U \subset\subset \Omega \setminus K$ be an open set such that $\mathrm{meas}((\Omega \setminus K) \setminus U)) < \delta$. As $U \subset\subset \Omega \setminus K_n$ for n large enough, we have also $\mathrm{meas}((\Omega \setminus K_n) \setminus U)) < \delta$. Then

$$\Big| \int_\Omega (\nabla u_n - \nabla u) \cdot \psi\, dx \Big| \le \Big| \int_U (\nabla u_n - \nabla u) \cdot \psi\, dx \Big| + c_1\varepsilon + c_2\varepsilon\,,$$

where c_1 is an upper bound for $\|\nabla u_n\|_{L^2(\Omega;\mathbb{R}^2)}$ and $c_2 := \|\nabla u\|_{L^2(\Omega;\mathbb{R}^2)}$. From the previous part of the lemma $\limsup_n |\int_\Omega (\nabla u_n - \nabla u) \cdot \psi\, dx| \le c_1\varepsilon + c_2\varepsilon$ and the conclusion follows from the arbitrariness of ε. ☐

Throughout the paper R denotes the rotation on $\mathbb{R}^2$ defined by $R(y_1, y_2) := (-y_2, y_1)$. In the next theorem we prove that every point of $\overline{\Omega}$ has an open neighbourhood U such that every solution u of (4.1) has a harmonic conjugate v on $(U \cap \Omega) \setminus K$ which is constant on each connected component of $\overline{U} \cap K$ and on each connected component of $\overline{U} \cap \partial_N\Omega$.

Theorem 4.2. *Let $K \in \mathcal{K}(\overline{\Omega})$, let u be a solution of problem (4.2), and let U be an open rectangle contained in Ω, or a rectangle as in the definition of the Lipschitz part of the boundary. Then there exists a function $v \in H^1(U \cap \Omega)$ such that $\nabla v = R\,\nabla u$ a.e. on $U \cap \Omega$. Moreover v is constant q.e. on each connected component of $\overline{U} \cap K$ and of $\overline{U} \cap \partial_N\Omega$.*

Proof. If $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(\varphi) \subset U \cap \Omega$, we have

$$\int_{U\cap\Omega} \nabla u\, \nabla\varphi\, dx = \int_{\Omega \setminus K} \nabla u\, \nabla\varphi\, dx = 0\,, \tag{4.6}$$

where the first equality follows from our convention $\nabla u = 0$ a.e. in K, while the second equality follows from (4.2), since $\varphi = 0$ on $\partial_D\Omega$. Equality (4.6) implies that $\mathrm{div}(\nabla u) = 0$ in $\mathcal{D}'(U \cap \Omega)$, hence $\mathrm{rot}(R\,\nabla u) = 0$ in $\mathcal{D}'(U \cap \Omega)$. As $U \cap \Omega$ is simply connected and has a Lipschitz boundary, there exists $v \in H^1(U \cap \Omega)$ such that $\nabla v = R\,\nabla u$ a.e. in $U \cap \Omega$.

Since $\nabla v = 0$ a.e. in $U \cap K$, the function v is constant q.e. on each connected open subset C of $U \cap K$, and, by (2.1), also on $C \cup \partial_L C$.

To prove that v is constant q.e. on each connected component of $\overline{U} \cap K$ we use an approximation argument. We write K as the intersection of a decreasing sequence (K_j) of compact subsets of $\overline{\Omega}$ such that $K \subset \mathrm{int}_{\overline{\Omega}} K_j$ for every j, where $\mathrm{int}_{\overline{\Omega}} K_j$ denotes the interior of K_j in the relative topology of $\overline{\Omega}$.

Note that u satisfies

$$\begin{cases} \displaystyle\int_{(U\cap\Omega)\setminus K} \nabla u \, \nabla z \, dx = 0 \\ \forall z \in L^{1,2}((U \cap \Omega)\setminus K)\,, \ z = 0 \quad \text{q.e. on } \partial(U \cap \Omega)\setminus K\,, \end{cases} \tag{4.7}$$

since every such function z can be extended to a function of $L^{1,2}(\Omega\setminus K)$ by setting $z = 0$ in $(\Omega\setminus U)\setminus K$. As $u \in L^{1,2}((U \cap \Omega)\setminus K_j)$, there exists a solution u_j to the problem

$$\begin{cases} u_j \in L^{1,2}((U \cap \Omega)\setminus K_j)\,, \quad u_j = u \quad \text{q.e. on } \partial(U \cap \Omega)\setminus K_j\,, \\ \displaystyle\int_{(U\cap\Omega)\setminus K_j} \nabla u_j \, \nabla z \, dx = 0 \\ \forall z \in L^{1,2}((U \cap \Omega)\setminus K_j)\,, \ z = 0 \quad \text{q.e. on } \partial(U \cap \Omega)\setminus K_j\,. \end{cases} \tag{4.8}$$

Using $u_j - u$ as test function in (4.8), we obtain that the norms $\|\nabla u_j\|_{L^2((U\cap\Omega)\setminus K_j)}$ are uniformly bounded. By Lemma 4.1, there exists $u^* \in L^{1,2}((U \cap \Omega)\setminus K)$, with $u^* = u$ q.e. on $\partial(U \cap \Omega)\setminus K$, such that, up to a subsequence, (∇u_j) converges to ∇u^* weakly in $L^2(U \cap \Omega; \mathbb{R}^2)$.

Taking $u_j - u^*$ as test function in (4.8), we get

$$\int_{U\cap\Omega} |\nabla u_j|^2 \, dx = \int_{U\cap\Omega} \nabla u_j \, \nabla u^* \, dx\,.$$

Passing to the limit we obtain that $\|\nabla u_j\|_{L^2(U\cap\Omega;\mathbb{R}^2)}$ converges to $\|\nabla u^*\|_{L^2(U\cap\Omega;\mathbb{R}^2)}$, hence ∇u_j converges to ∇u^* strongly in $L^2(U \cap \Omega; \mathbb{R}^2)$.

Let us prove that

$$\nabla u^* = \nabla u \quad \text{ a.e. in } (U \cap \Omega)\setminus K\,. \tag{4.9}$$

By the uniqueness of the gradients of the solutions, it is enough to prove that u^* is a solution of (4.7).

Let $z \in L^{1,2}((U \cap \Omega)\setminus K)$ with $z = 0$ q.e. on $\partial(U \cap \Omega)\setminus K$. As $z \in L^{1,2}((U \cap \Omega)\setminus K_j)$ and $z = 0$ q.e. on $\partial(U \cap \Omega)\setminus K_j$, we can use z as test function in (4.8). Then passing to the limit as $j \to \infty$ we obtain (4.7), and the proof of (4.9) is complete.

By the first part of the proof, there exist a function $v_j \in H^1(U \cap \Omega)$, such that $\nabla v_j = R\,\nabla u_j$ a.e. on $U \cap \Omega$. Let K^0 be a connected component of $\overline{U} \cap K$. It is easy to see that there exists a connected component C of the interior of $U \cap K_j$ such that $K^0 \subset C \cup \partial_L C$ (this is trivial if $K^0 \subset U \cap \Omega$, and follows from the regularity of $\partial(U \cap \Omega)$ if K^0 meets $\partial(U \cap \Omega)$). As v_j is constant q.e. on $C \cup \partial_L C$, we obtain that v_j is constant q.e. on K^0.

We may assume that $\int_{U\cap\Omega} v_j \, dx = 0$ for every j. Since $\nabla v_j = R\,\nabla u_j$ a.e. on $U \cap \Omega$ we deduce that (∇v_j) converges to $R\,\nabla u$ strongly in $L^2(U \cap \Omega; \mathbb{R}^2)$, and by the Poincaré inequality (v_j) converges strongly in $H^1(U \cap \Omega)$ to a function v which satisfies $\nabla v = R\,\nabla u$ a.e. on $U \cap \Omega$. As v_j is constant q.e. on K^0, we conclude that v is constant q.e. on K^0.

To prove that v is constant q.e. on each connected component of $\overline{U} \cap \partial_N \Omega$, it is enough to show that v is constant q.e. on $V \cap \partial_N \Omega$ whenever $V \subset U$ is a rectangle as in the definition of the Lipschitz part of the boundary and $V \cap \partial\Omega = V \cap \partial_N \Omega$. Let $\psi \in L^2(V; \mathbb{R}^2)$ be the vector-field defined by $\psi = \nabla u$ a.e. in $V \cap \Omega$ and $\psi = 0$ a.e. in $V\setminus\Omega$. As at the beginning of the proof, it is easy to see that $\mathrm{div}(\psi) = 0$ in $\mathcal{D}'(V)$, hence $\mathrm{rot}(R\psi) = 0$ in $\mathcal{D}'(V)$. Then there exists a function $z \in H^1(V)$ such that $\nabla z = R\psi$ a.e in V. As

$\nabla z = 0$ a.e. in the connected set $V\backslash\overline{\Omega}$, using (2.1) we obtain that z is constant q.e. in $V\backslash\Omega$. As $\nabla z = R\,\nabla u = \nabla v$ a.e. in the connected set $V\cap\Omega$, using (2.1) we obtain that $z - v$ is constant q.e. in $V\cap\overline{\Omega}$. From these facts we deduce that v is constant q.e. on $V\cap\partial\Omega = V\cap\partial_N\Omega$. □

Theorem 4.3. *Let K be a locally connected compact subset of $\overline{\Omega}$ and let $u\in L^{1,2}(\Omega\backslash K)$. Assume that for every $x\in\overline{\Omega}$ there exist an open neighbourhood U of x in $\mathbb{R}^2$ and a function $v\in H^1(U\cap\Omega)$, with $\nabla v = R\,\nabla u$ a.e. in $U\cap\Omega$, such that v is constant q.e. on each connected component of $U\cap K$ and of $U\cap\partial_N\Omega$. Then u is a solution of problem (4.2).*

Proof. By a standard localization argument, it is enough to prove that for every $x\in\overline{\Omega}$ there exists an open neighbourhood V of x in $\mathbb{R}^2$ such that

$$\begin{cases} \displaystyle\int_{(V\cap\Omega)\backslash K} \nabla u\,\nabla z\,dx = 0 \\ \forall z\in L^{1,2}((V\cap\Omega)\backslash K)\,,\ z=0 \quad \text{q.e. on } (V\cap\partial_D\Omega)\backslash K\,, \quad \operatorname{supp}(z)\subset\subset V\,. \end{cases} \tag{4.10}$$

For every $x\in\overline{\Omega}$ let U be the neighbourhood given in the statement of the theorem. Taking, if necessary, a smaller neighbourhood, we may assume that $U\cap\Omega$ has a Lipschitz boundary and that v is constant q.e. on the closure of each connected component of $U\cap K$ and of $U\cap\partial_N\Omega$. Let V be an arbitrary open neighbourhood of x in $\mathbb{R}^2$ with $V\subset\subset U$. Since K is locally connected, the connected components of $U\cap K$ are open in K, so that only a finite number of them meets $\overline{V}\cap K$. Similarly, only a finite number of connected components of $U\cap\partial_N\Omega$ meets $\overline{V}\cap\partial_N\Omega$. Using Proposition 2.5 it is easy to prove that there exist a finite family $\widehat{K}^1,\dots,\widehat{K}^m$ of pairwise disjoint compact sets and a family of distinct constants $c^1,\dots,c^m$ such that

$$\overline{V}\cap(K\cup\partial_N\Omega) = \widehat{K}^1\cup\dots\cup\widehat{K}^m$$

and $v = c^i$ q.e. on $\widehat{K}^i$ for $i = 1,\dots,m$.

We now apply [25, Theorem 4.5] (to a suitable extension of $v|_{V\cap\Omega}$) and construct a sequence of functions $v_n\in C^\infty(\mathbb{R}^2)$, converging to v in $H^1(V\cap\Omega)$, such that $v_n = c^i$ in a neighbourhood U_n^i of each $\widehat{K}^i$.

Let $z\in L^{1,2}((V\cap\Omega)\backslash K)$, with compact support in V, such that $z=0$ q.e. on $(V\cap\partial_D\Omega)\backslash K$, let φ_n^i be functions in $C_c^\infty(\mathbb{R}^2)$, with $\operatorname{supp}(\varphi_n^i)\subset U_n^i$, such that $\varphi_n^i = 1$ in a neighbourhood of $\widehat{K}^i$, and let $\psi_n := 1-\sum_i\varphi_n^i$. By Proposition 2.2 the function $z\,\psi_n$ belongs to $H^1((V\cap\Omega)\backslash K)$, and by [25, Theorem 4.5] it belongs to $H_0^1((V\cap\Omega)\backslash K)$.

Since $\psi_n = 1$ where $R\,\nabla v_n \neq 0$, we have

$$\int_{(V\cap\Omega)\backslash K} R\,\nabla v_n\,\nabla z\,dx = \int_{(V\cap\Omega)\backslash K} R\,\nabla v_n\,\nabla(z\,\psi_n)\,dx = 0\,,$$

where the last equality follows from the fact that $\operatorname{div}(R\,\nabla v_n) = 0$ in $\mathbb{R}^2$ and $z\,\psi_n\in H_0^1((V\cap\Omega)\backslash K)$. Passing to the limit as $n\to\infty$, we obtain

$$\int_{(V\cap\Omega)\backslash K} \nabla u\,\nabla z\,dx = -\int_{(V\cap\Omega)\backslash K} R\,\nabla v\,\nabla z\,dx = 0\,,$$

showing that u is a solution of (4.10). □

5. CONVERGENCE OF MINIMIZERS

In this section we prove the convergence of the minimum points of problems (4.4) corresponding to a sequence (K_n) in $\mathcal{K}_m^\lambda(\overline{\Omega})$ which converges in the Hausdorff metric.

Theorem 5.1. *Let $m \geq 1$ and $\lambda \geq 0$, let (K_n) be a sequence in $\mathcal{K}_m^\lambda(\overline{\Omega})$ which converges to K in the Hausdorff metric, and let (g_n) be a sequence in $H^1(\Omega)$ which converges to g strongly in $H^1(\Omega)$. Let u_n be a solution of the minimum problem*

$$\min_{v \in \mathcal{V}(g_n, K_n)} \int_{\Omega \setminus K_n} |\nabla v|^2 \, dx \,, \tag{5.1}$$

and let u be a solution of the minimum problem

$$\min_{v \in \mathcal{V}(g, K)} \int_{\Omega \setminus K} |\nabla v|^2 \, dx \,, \tag{5.2}$$

where $\mathcal{V}(g_n, K_n)$ and $\mathcal{V}(g, K)$ are defined by (4.5). Then $\nabla u_n \to \nabla u$ strongly in $L^2(\Omega; \mathbb{R}^2)$.

The following lemma is crucial in the proof of Theorem 5.1.

Lemma 5.2. *Let (K_n) be a sequence in $\mathcal{K}_1(\overline{\Omega})$ which converges to K in the Hausdorff metric, and let (v_n) be a sequence in $H^1(\Omega)$ which converges to v weakly in $H^1(\Omega)$. Assume that $v_n = 0$ q.e. on K_n for every n. Then $v = 0$ q.e. on K.*

Proof. Let us fix an open ball B containing $\overline{\Omega}$. Using the same extension operator we can construct extensions of v_n and v, still denoted by v_n and v, such that $v_n, v \in H_0^1(B)$, $v_n \rightharpoonup v$ weakly in $H^1(B)$.

Given an open set $A \subset B$, any function $z \in H_0^1(A)$ will be extended to a function $z \in H_0^1(B)$ by setting $z := 0$ q.e. in $\overline{B} \setminus A$. By [25, Theorem 4.5] we have

$$H_0^1(A) = \{ z \in H^1(B) : z = 0 \quad \text{q.e. on } \overline{B} \setminus A \} \,. \tag{5.3}$$

Since the complement of $B \setminus K_n$ has two connected components, from the results of [37] and [10] we deduce that, for every $f \in L^2(B)$, the solutions z_n of the Dirichlet problems

$$z_n \in H_0^1(B \setminus K_n) \qquad \Delta z_n = f \quad \text{in } B \setminus K_n$$

converge strongly in $H_0^1(B)$ to the solution z of the Dirichlet problem

$$z \in H_0^1(B \setminus K) \qquad \Delta z = f \quad \text{in } B \setminus K \,.$$

This implies (see, e.g., [1, Theorem 3.33]) that, in the space $H_0^1(B)$, the subspaces $H_0^1(B \setminus K_n)$ converge to the subspace $H_0^1(B \setminus K)$ in the sense of Mosco (see [29, Definition 1.1]).

Since $v_n \in H_0^1(B \setminus K_n)$ by (5.3), and $v_n \rightharpoonup v$ weakly in $H^1(B)$, from the convergence in the sense of Mosco we deduce that $v \in H_0^1(B \setminus K)$, hence $v = 0$ q.e. on K by (5.3). ☐

Proof of Theorem 5.1. Note that u is a minimum point of (5.2) if and only if u satisfies (4.2) and (4.3); analogously, u_n is a minimum point of (5.1) if and only if u_n satisfies (4.2) and (4.3) with K and g replaced by K_n and g_n.

Taking $u_n - g_n$ as test function in the equation satisfied by u_n, we prove that the sequence (∇u_n) is bounded in $L^2(\Omega; \mathbb{R}^2)$. By Lemma 4.1, there exists a function $u^* \in L^{1,2}(\Omega \setminus K)$, with $u^* = g$ q.e. on $\partial_D \Omega \setminus K$, such that, passing to a subsequence, $\nabla u_n \rightharpoonup \nabla u^*$ weakly in $L^2(\Omega; \mathbb{R}^2)$.

We will prove that

$$\nabla u^* = \nabla u \quad \text{ a.e. in } \Omega \setminus K \,. \tag{5.4}$$

As the limit does not depend on the subsequence, this implies that the whole sequence (∇u_n) converges to ∇u weakly in $L^2(\Omega; \mathbb{R}^2)$. Taking again $u_n - g_n$ and $u - g$ as test functions in the equations satisfied by u_n and u, we obtain

$$\int_\Omega |\nabla u_n|^2 dx = \int_\Omega \nabla u_n \nabla g_n \, dx \,, \qquad \int_\Omega |\nabla u|^2 dx = \int_\Omega \nabla u \, \nabla g \, dx \,.$$

As $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega, \mathbb{R}^2)$ and $\nabla g_n \to \nabla g$ strongly in $L^2(\Omega, \mathbb{R}^2)$, from the previous equalities we obtain that $\|\nabla u_n\|_{L^2(\Omega, \mathbb{R}^2)}$ converges to $\|\nabla u\|_{L^2(\Omega, \mathbb{R}^2)}$, which implies the strong convergence of the gradients in $L^2(\Omega, \mathbb{R}^2)$.

By the uniqueness of the gradients of the solutions, to prove (5.4) it is enough to show that u^* is a solution of (4.2). This will be obtained by using Theorem 4.3. First of all we note that $K \in \mathcal{K}_m^\lambda(\overline{\Omega})$ by Corollary 3.3, and therefore K is locally connected (see, e.g., [12, Lemma 1]).

Let us fix $x \in \overline{\Omega}$ and an open rectangle V containing x. If $x \in \Omega$, we assume that $V \subset \Omega$. If $x \in \partial\Omega$, we assume that V is as in the definition of the Lipschitz part of the boundary. Let U be an open neighbourhood of x in $\mathbb{R}^2$ such that $U \subset\subset V$. We will prove that there exists a function $v \in H^1(U \cap \Omega)$, with $\nabla v = R\,\nabla u^*$ a.e. in $U \cap \Omega$, such that v is constant q.e. on each connected component of $U \cap K$ and of $U \cap \partial_N\Omega$. By Theorem 4.3 this implies that u^* satisfies (4.2).

Let $\delta := \mathrm{dist}(U, \partial V)$. Let us prove that there are at most $m + \lambda/\delta$ connected components C of $\overline{V} \cap K_n$ which meet $U \cap K_n$. Indeed, if C meets also ∂V, then $\mathcal{H}^1(C) \geq \delta$ (since C connects a point in U with a point in ∂V), so that the number of these components can not exceed λ/δ. On the other hand, it is easy to see that the other connected components of $\overline{V} \cap K_n$ are also connected components of K_n, thus their number can not exceed m.

Let $K_n^1, \dots, K_n^{k_n}$ be the connected components of $\overline{V} \cap K_n$ which meet $U \cap K_n$. As $k_n \leq m + \lambda/\delta$, passing to a subsequence we may assume that $k_n = k$ for every n, and that $K_n^1 \to \widehat{K}^1$, ..., $K_n^k \to \widehat{K}^k$ in the Hausdorff metric, where $\widehat{K}^1, \dots, \widehat{K}^k$ are compact and connected. Arguing as in the proof of Corollary 3.3, we obtain that

$$U \cap K \subset \widehat{K}^1 \cup \dots \cup \widehat{K}^k \,. \tag{5.5}$$

Let v_n be the harmonic conjugate of u_n in $V \cap \Omega$ given by Theorem 4.2. Then $\nabla v_n = R\,\nabla u_n$ a.e. in $V \cap \Omega$. We may assume that $\int_{V\cap\Omega} v_n\,dx = 0$ for every n. Since $\nabla v_n = R\,\nabla u_n$ a.e. on $V \cap \Omega$, we deduce that (∇v_n) converges to $R\,\nabla u^*$ weakly in $L^2(V \cap \Omega; \mathbb{R}^2)$, and by the Poincaré inequality (v_n) converges weakly in $H^1(V \cap \Omega)$ to a function v which satisfies $\nabla v = R\,\nabla u^*$ a.e. on $V \cap \Omega$.

Let us prove that for every $i = 1, \dots, k$ there exists a constant c^i such that $v = c^i$ q.e. on $\widehat{K}^i$. This is trivial when $\widehat{K}^i$ reduces to one point. If $\widehat{K}^i$ has more than one point, then $\liminf_n \mathrm{diam}(K_n^i) > 0$; since the sets K_n^i are connected, we obtain also $\liminf_n \mathrm{cap}(K_n^i) > 0$. As $v_n = c_n^i$ q.e. on K_n^i for suitable constants c_n^i, using the Poincaré inequality (see, e.g., [38, Corollary 4.5.3]) it follows that $(v_n - c_n^i)$ is bounded in $H^1(V \cap \Omega)$, hence the sequence (c_n^i) is bounded, and therefore, passing to a subsequence, we may assume that $c_n^i \to c^i$ for a suitable constant c^i. Then $(v_n - c_n^i)$ converges to $v - c^i$ weakly in $H^1(V \cap \Omega)$, and by Lemma 5.2 we conclude that $v = c^i$ q.e. on $\widehat{K}^i$.

By Proposition 2.5, if $\widehat{K}^i \cap \widehat{K}^j \neq \emptyset$, then v is constant q.e. on $\widehat{K}^i \cup \widehat{K}^j$. By (5.5) this implies that v is constant q.e. on each connected component of $U \cap K$.

On the other hand, every v_n is constant q.e. on each connected component of $V \cap \partial_N\Omega$. Since $v_n \rightharpoonup v$ weakly in $H^1(V \cap \Omega)$, a sequence of convex combinations of the functions v_n converges to v strongly in $H^1(V \cap \Omega)$, and we conclude that v is constant q.e. on each connected component of $V \cap \partial_N\Omega$, hence on each connected component of $U \cap \partial_N\Omega$.

Therefore u^* satisfies all hypotheses of Theorem 4.3, which implies that u^* is a solution of problem (4.2). $\square$

6. COMPACT VALUED INCREASING FUNCTIONS

In this section we consider increasing functions $K\colon [0,1] \to \mathcal{K}(\overline{\Omega})$, i.e., we assume that $K(s) \subset K(t)$ for $s < t$. The following proposition extends to compact valued increasing functions a well known result about the continuity of real valued monotone functions.

Proposition 6.1. *Let* $K\colon[0,1]\to\mathcal{K}(\overline{\Omega})$ *be an increasing function, and let* $K^-\colon(0,1]\to\mathcal{K}(\overline{\Omega})$ *and* $K^+\colon[0,1)\to\mathcal{K}(\overline{\Omega})$ *be the functions defined by*

$$K^-(t):=\operatorname{cl}\big(\textstyle\bigcup_{s<t}K(s)\big)\qquad \textit{for } 0<t\le 1\,, \tag{6.1}$$

$$K^+(t):=\textstyle\bigcap_{s>t}K(s)\qquad \textit{for } 0\le t<1\,, \tag{6.2}$$

where cl *denotes the closure. Then*

$$K^-(t)\subset K(t)\subset K^+(t)\qquad \textit{for } 0<t<1\,. \tag{6.3}$$

Let Θ *be the set of points* $t\in(0,1)$ *such that* $K^-(t)=K^+(t)$. *Then* $[0,1]\backslash\Theta$ *is at most countable, and* $K(t_n)\to K(t)$ *in the Hausdorf metric for every* $t\in\Theta$ *and every sequence* (t_n) *in* $[0,1]$ *converging to* t.

To prove Proposition 6.1 we use the following result, which extends another well known property of real valued monotone functions.

Lemma 6.2. *Let* K_1, $K_2\colon[0,1]\to\mathcal{K}(\overline{\Omega})$ *be two increasing functions such that*

$$K_1(s)\subset K_2(t)\qquad\textit{and}\qquad K_2(s)\subset K_1(t) \tag{6.4}$$

for every s, $t\in[0,1]$ *with* $s<t$. *Let* Θ *be the set of points* $t\in[0,1]$ *such that* $K_1(t)=K_2(t)$. *Then* $[0,1]\backslash\Theta$ *is at most countable.*

Proof. For $i=1,2$, consider the functions $f_i\colon\overline{\Omega}\times[0,1]\to\mathbb{R}$ defined by $f_i(x,t):=\operatorname{dist}(x,K_i(t))$, with the convention that $\operatorname{dist}(x,\emptyset)=\operatorname{diam}(\Omega)$. Then the functions $f_i(\cdot,t)$ are Lipschitz continuous with constant 1 for every $t\in[0,1]$, and the functions $f_i(x,\cdot)$ are non-increasing for every $x\in\overline{\Omega}$.

Let D be a countable dense subset of $\overline{\Omega}$. For every $x\in D$ there exists a countable set $N_x\subset[0,1]$ such that $f_i(x,\cdot)$ are continuous at every point of $[0,1]\backslash N_x$. By (6.4) we have $f_1(x,s)\ge f_2(x,t)$ and $f_2(x,s)\ge f_1(x,t)$ for every $x\in\overline{\Omega}$ and every s, $t\in[0,1]$ with $s<t$. This implies that $f_1(x,t)=f_2(x,t)$ for every $x\in D$ and every $t\in[0,1]\backslash N_x$. Let N be the countable set defined by $N:=\bigcup_{x\in D}N_x$, and let $t\in[0,1]\backslash N$. Then $f_1(x,t)=f_2(x,t)$ for every $x\in D$, and, by continuity, for every $x\in\overline{\Omega}$, which yields $K_1(t)=K_2(t)$. This proves that $[0,1]\backslash N\subset\Theta$. □

Proof of Proposition 6.1. It is clear that K^+ and K^- are increasing and satisfy (6.4). Therefore $[0,1]\backslash\Theta$ is at most countable by Lemma 6.2.

Let us fix $t\in\Theta$ and a sequence (t_n) in $[0,1]$ converging to t. By the Compactness Theorem 3.1 we may assume that $K(t_n)$ converges in the Hausdorff metric to a set K^*. For every s_1, $s_2\in[0,1]$, with $s_1<t<s_2$, we have $K(s_1)\subset K(t_n)\subset K(s_2)$ for n large enough, hence $K(s_1)\subset K^*\subset K(s_2)$. As K^* is closed this implies $K^-(t)\subset K^*\subset K^+(t)$, therefore $K^*=K(t)$ by (6.3) and by the definition of Θ. □

The following result is the analogue of the Helly theorem for compact valued increasing functions.

Theorem 6.3. *Let* (K_n) *be a sequence of increasing functions from* $[0,1]$ *into* $\mathcal{K}(\overline{\Omega})$. *Then there exist a subsequence, still denoted by* (K_n), *and an increasing function* $K\colon[0,1]\to\mathcal{K}(\overline{\Omega})$, *such that* $K_n(t)\to K(t)$ *in the Hausdorff metric for every* $t\in[0,1]$.

Proof. Let D be a countable dense subset of $(0,1)$. Using a diagonal argument, we find a subsequence, still denoted by (K_n), and an increasing function $K\colon D\to\mathcal{K}(\overline{\Omega})$, such that $K_n(t)\to K(t)$ in the Hausdorff metric for every $t\in D$. Let $K^-\colon(0,1]\to\mathcal{K}(\overline{\Omega})$ and $K^+\colon[0,1)\to\mathcal{K}(\overline{\Omega})$ be the increasing functions defined by

$$K^-(t):=\operatorname{cl}\Big(\bigcup_{s<t,\ s\in D}K(s)\Big)\qquad\text{for } 0<t\le 1\,,$$

$$K^+(t):=\bigcap_{s>t,\ s\in D}K(s)\qquad\text{for } 0\le t<1\,,$$

where cl denotes the closure. Let Θ be the set of points $t \in [0,1]$ such that $K^-(t) = K^+(t)$. As K^- and K^+ satisfy (6.4), by Lemma 6.2 the set $[0,1] \backslash \Theta$ is at most countable.

Since $K^-(t) \subset K(t) \subset K^+(t)$ for every $t \in D$, we have $K(t) = K^-(t) = K^+(t)$ for every $t \in \Theta \cap D$. For every $t \in \Theta \backslash D$ we define $K(t) := K^-(t) = K^+(t)$. To prove that $K_n(t) \to K(t)$ for a given $t \in \Theta \backslash D$, by the Compactness Theorem 3.1 we may assume that $K_n(t)$ converges in the Hausdorff metric to a set K^*. For every s_1, $s_2 \in D$, with $s_1 < t < s_2$, by monotonicity we have $K(s_1) \subset K^* \subset K(s_2)$. As K^* is closed, this implies $K^-(t) \subset K^* \subset K^+(t)$, therefore $K_n(t) \to K(t)$ by the definitions of Θ and $K(t)$.

Since $[0,1] \backslash (\Theta \cup D)$ is at most countable, by a diagonal argument we find a further subsequence, still denoted by (K_n), and a function $K\colon [0,1] \backslash (\Theta \cup D) \to \mathcal{K}(\overline{\Omega})$, such that $K_n(t) \to K(t)$ in the Hausdorff metric for every $t \in [0,1] \backslash (\Theta \cup D)$.

Therefore $K_n(t) \to K(t)$ for every $t \in [0,1]$, and this implies that K is increasing on $[0,1]$. ☐

For every compact set K in $\mathbb{R}^2$ and every $g \in L^{1,2}(\Omega \backslash K)$ we define

$$\mathcal{E}(g,K) := \min_{v \in \mathcal{V}(g,K)} \Big\{ \int_{\Omega \backslash K} |\nabla v|^2 dx + \mathcal{H}^1(K) \Big\}, \tag{6.5}$$

where $\mathcal{V}(g,K)$ is the set introduced in (4.5).

Given a Hilbert space X, we recall that $AC([0,1];X)$ is the space of all absolutely continuous functions defined in $[0,1]$ with values in X. For the main properties of these functions we refer, e.g., to [7, Appendix]. Given $g \in AC([0,1];X)$, the time derivative of g, which exists a.e. in $[0,1]$, is denoted by $\dot g$. It is well-known that $\dot g$ is a Bochner integrable function with values in X.

The following result will be crucial in the next section.

Theorem 6.4. *Let $m \geq 1$, let $g \in AC([0,1]; H^1(\Omega))$, and let $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ be an increasing function. Suppose that the function $t \mapsto \mathcal{E}(g(t),K(t))$ is absolutely continuous on $[0,1]$. Then the following conditions are equivalent:*

(a) $$\frac{d}{ds}\mathcal{E}(g(t),K(s))\Big|_{s=t} = 0 \qquad \text{for a.e. } t \in [0,1],$$

(b) $$\frac{d}{dt}\mathcal{E}(g(t),K(t)) = 2(\nabla u(t) | \nabla \dot g(t)) \qquad \text{for a.e. } t \in [0,1],$$

where $u(t)$ is a solution of the minimum problem (6.5) which defines $\mathcal{E}(g(t),K(t))$, and $(\cdot|\cdot)$ denotes the scalar product in $L^2(\Omega;\mathbb{R}^2)$.

To prove Theorem 6.4 we need the following lemmas.

Lemma 6.5. *Let $K \in \mathcal{K}^f(\overline{\Omega})$ and let $F\colon H^1(\Omega) \to \mathbb{R}$ be defined by $F(g) = \mathcal{E}(g,K)$ for every $g \in H^1(\Omega)$. Then F is of class C^1 and for every $g,h \in H^1(\Omega)$ we have*

$$dF(g)\,h = 2\int_{\Omega \backslash K} \nabla u_g \nabla h \, dx, \tag{6.6}$$

where u_g is a solution of the minimum problem (6.5) which defines $\mathcal{E}(g,K)$.

Proof. Since u_g is a solution of problem (4.2) which satisfies the boundary condition (4.3), by linearity for every $t \in \mathbb{R}$ we have $\nabla u_{g+th} = \nabla u_g + t\nabla u_h$ a.e. in Ω, hence

$$F(g+th) - F(g) = \int_{\Omega \backslash K} |\nabla u_g + t\nabla u_h|^2 \, dx - \int_{\Omega \backslash K} |\nabla u_g|^2 \, dx =$$
$$= 2t \int_{\Omega \backslash K} \nabla u_g \nabla u_h \, dx + t^2 \int_{\Omega \backslash K} |\nabla u_h|^2 \, dx = 2t \int_{\Omega \backslash K} \nabla u_g \nabla h \, dx + t^2 \int_{\Omega \backslash K} |\nabla u_h|^2 \, dx,$$

where the last equality is deduced from (4.2). Dividing by t and letting t tend to 0 we obtain (6.6). The continuity of $g \mapsto \nabla u_g$ implies that F is of class C^1. ☐

Let us consider now the case of time dependent compact sets $K(t)$.

Lemma 6.6. *Let $m \geq 1$ and $\lambda \geq 0$, let $K\colon [0,1] \to \mathcal{K}^\lambda_m(\overline{\Omega})$ be a function, and let $F\colon H^1(\Omega)\times[0,1] \to \mathbb{R}$ be defined by $F(g,t) = \mathcal{E}(g,K(t))$. Then the differential d_1F of F with respect to g is continuous at every point $(g,t) \in H^1(\Omega)\times[0,1]$ such that $K(s) \to K(t)$ in the Hausdorff metric as $s \to t$.*

Proof. It is enough to apply Lemma 6.5 and Theorem 5.1. □

To deal with the dependence on t of both arguments we need the following result.

Lemma 6.7. *Let X be a Hilbert space, let $g \in AC([0,1];X)$, and let $F\colon X\times[0,1] \to \mathbb{R}$ be a function such that $F(\cdot,t) \in C^1(X)$ for every $t \in [0,1]$, with differential denoted by $d_1F(\cdot,t)$. Let $t_0 \in [0,1]$, let $\psi(t) := F(g(t),t)$, and let $\psi_0(t) := F(g(t_0),t)$. Assume that t_0 is a differentiability point of ψ and g and a Lebesgue point of $\dot g$, and that d_1F is continuous at $(g(t_0),t_0)$. Then ψ_0 is differentiable at t_0 and*

$$\dot\psi_0(t_0) = \dot\psi(t_0) - d_1F(g(t_0),t_0)\,\dot g(t_0)\,.$$

Proof. For every $t \in [0,1]$ we have

$$\begin{aligned}\psi_0(t) - \psi_0(t_0) = {}& F(g(t_0),t) - F(g(t),t) + \psi(t) - \psi(t_0) = \\ = {}& \int_t^{t_0} d_1F(g(s),t)\,\dot g(s)\,ds + \psi(t) - \psi(t_0)\,.\end{aligned}$$

The conclusion follows dividing by $t - t_0$ and taking the limit as $t \to t_0$. □

Proof of Theorem 6.4. Let $F\colon H^1(\Omega)\times[0,1] \to \mathbb{R}$ be defined by $F(g,t) = \mathcal{E}(g,K(t))$. By Proposition 6.1 and Lemma 6.6 d_1F is continuous in (g,t) for a.e. $t \in [0,1]$ and every $g \in H^1(\Omega)$. By Lemmas 6.5 and 6.7

$$\frac{d}{ds}\mathcal{E}(g(t),K(s))\Big|_{s=t} = \frac{d}{dt}\mathcal{E}(g(t),K(t)) - 2(\nabla u(t)|\nabla \dot g(t)) \quad \text{for a.e. } t \in [0,1]\,. \tag{6.7}$$

The equivalence between (a) and (b) is now obvious. □

7. IRREVERSIBLE QUASI-STATIC EVOLUTION

In this section we prove the main result of the paper.

Theorem 7.1. *Let $m \geq 1$, let $g \in AC([0,1];H^1(\Omega))$, and let $K_0 \in \mathcal{K}^f_m(\overline{\Omega})$. Then there exists a function $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ such that*

(a) $$K_0 \subset K(s) \subset K(t) \ \textit{for}\ 0 \leq s \leq t \leq 1\,,$$

(b) $$\mathcal{E}(g(0),K(0)) \leq \mathcal{E}(g(0),K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}),\ \ K \supset K_0\,,$$

(c) $$\textit{for}\ 0 \leq t \leq 1 \quad \mathcal{E}(g(t),K(t)) \leq \mathcal{E}(g(t),K) \quad \forall K \in \mathcal{K}^f_m(\overline{\Omega}),\ \ K \supset K(t)\,,$$

(d) $$t \mapsto \mathcal{E}(g(t),K(t)) \ \textit{is absolutely continuous on}\ [0,1]\,,$$

(e) $$\frac{d}{ds}\mathcal{E}(g(t),K(s))\Big|_{s=t} = 0 \quad \textit{for a.e.}\ t \in [0,1]\,.$$

Moreover every function $K\colon [0,1] \to \mathcal{K}^f_m(\overline{\Omega})$ which satisfies (a)–(e) satisfies also

(f) $$\frac{d}{dt}\mathcal{E}(g(t),K(t)) = 2(\nabla u(t)|\nabla \dot g(t)) \quad \textit{for a.e.}\ t \in [0,1]\,,$$

where $u(t)$ is a solution of the minimum problem (6.5) which defines $\mathcal{E}(g(t),K(t))$.

Here and in the rest of the section $(\cdot|\cdot)$ and $\|\cdot\|$ denote the scalar product and the norm in $L^2(\Omega;\mathbb{R}^2)$.

Theorem 7.1 will be proved by a time discretization process. Given $\delta > 0$, let N_δ be the largest integer such that $\delta N_\delta \le 1$; for $i \ge 0$ let $t_i^\delta := i\delta$ and, for $0 \le i \le N_\delta$, let $g_i^\delta := g(t_i^\delta)$. We define K_i^δ, inductively, as a solution of the minimum problem

$$\min_K \left\{ \mathcal{E}(g_i^\delta, K) : K \in \mathcal{K}_m^f(\overline{\Omega}),\ K \supset K_{i-1}^\delta \right\}, \tag{7.1}$$

where we set $K_{-1}^\delta := K_0$.

Lemma 7.2. *There exists a solution of the minimum problem (7.1).*

Proof. By hypothesis $K_{-1}^\delta := K_0 \in \mathcal{K}_m^f(\overline{\Omega})$. Assume by induction that $K_{i-1}^\delta \in \mathcal{K}_m^f(\overline{\Omega})$ and let λ be a constant such that $\lambda > \mathcal{E}(g_i^\delta, K_{i-1}^\delta)$. Consider a minimizing sequence (K_n) of problem (7.1). We may assume that $K_n \in \mathcal{K}_m^\lambda(\overline{\Omega})$ for every n. By the Compactness Theorem 3.1, passing to a subsequence, we may assume that (K_n) converges in the Hausdorff metric to some compact set K containing K_{i-1}^δ. For every n let u_n be a solution of the minimum problem (6.5) which defines $\mathcal{E}(g_i^\delta, K_n)$. By Theorem 5.1 (∇u_n) converges strongly in $L^2(\Omega;\mathbb{R}^2)$ to ∇u, where u is a solution of the minimum problem (6.5) which defines $\mathcal{E}(g_i^\delta, K)$. By Corollary 3.3 we have $K \in \mathcal{K}_m(\overline{\Omega})$ and $\mathcal{H}^1(K) \le \liminf_n \mathcal{H}^1(K_n) \le \lambda$, hence $K \in \mathcal{K}_m^\lambda(\overline{\Omega})$. As $\|\nabla u\| = \lim_n \|\nabla u_n\|$, we conclude that $\mathcal{E}(g_i^\delta, K) \le \liminf_n \mathcal{E}(g_i^\delta, K_n)$. Since (K_n) is a minimizing sequence, this proves that K is a solution of the minimum problem (7.1). □

We define now the step functions g_δ, K_δ, and u_δ on $[0,1]$ by setting $g_\delta(t) := g_i^\delta$, $K_\delta(t) := K_i^\delta$, and $u_\delta(t) := u_i^\delta$ for $t_i^\delta \le t < t_{i+1}^\delta$, where u_i^δ is a solution of the minimum problem (6.5) which defines $\mathcal{E}(g_i^\delta, K_i^\delta)$.

Lemma 7.3. *There exists a positive function $\rho(\delta)$, converging to zero as $\delta \to 0$, such that*

$$\|\nabla u_j^\delta\|^2 + \mathcal{H}^1(K_j^\delta) \le \|\nabla u_i^\delta\|^2 + \mathcal{H}^1(K_i^\delta) + 2\int_{t_i^\delta}^{t_j^\delta} (\nabla u_\delta(t)|\nabla \dot g(t))\,dt + \rho(\delta) \tag{7.2}$$

for $0 \le i < j \le N_\delta$.

Proof. Let us fix an integer r with $i \le r < j$. From the absolute continuity of g we have

$$g_{r+1}^\delta - g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \dot g(t)\,dt\,,$$

where the integral is a Bochner integral for functions with values in $H^1(\Omega)$. This implies that

$$\nabla g_{r+1}^\delta - \nabla g_r^\delta = \int_{t_r^\delta}^{t_{r+1}^\delta} \nabla \dot g(t)\,dt\,, \tag{7.3}$$

where the integral is a Bochner integral for functions with values in $L^2(\Omega;\mathbb{R}^2)$.

As $u_r^\delta + g_{r+1}^\delta - g_r^\delta \in L^{1,2}(\Omega \setminus K_r^\delta)$ and $u_r^\delta + g_{r+1}^\delta - g_r^\delta = g_{r+1}^\delta$ q.e. on $\partial_D\Omega \setminus K_r^\delta$, we have

$$\mathcal{E}(g_{r+1}^\delta, K_r^\delta) \le \|\nabla u_r^\delta + \nabla g_{r+1}^\delta - \nabla g_r^\delta\|^2 + \mathcal{H}^1(K_r^\delta)\,. \tag{7.4}$$

By the minimality of u_{r+1}^δ and by (7.1) we have

$$\|\nabla u_{r+1}^\delta\|^2 + \mathcal{H}^1(K_{r+1}^\delta) = \mathcal{E}(g_{r+1}^\delta, K_{r+1}^\delta) \le \mathcal{E}(g_{r+1}^\delta, K_r^\delta)\,. \tag{7.5}$$

From (7.3), (7.4), and (7.5) we obtain

$$\begin{aligned}
\|\nabla u_{r+1}^\delta\|^2 + \mathcal{H}^1(K_{r+1}^\delta) &\le \|\nabla u_r^\delta + \nabla g_{r+1}^\delta - \nabla g_r^\delta\|^2 + \mathcal{H}^1(K_r^\delta) \le \\
&\le \|\nabla u_r^\delta\|^2 + \mathcal{H}^1(K_r^\delta) + 2\int_{t_r^\delta}^{t_{r+1}^\delta} (\nabla u_r^\delta|\nabla \dot g(t))\,dt + \Big(\int_{t_r^\delta}^{t_{r+1}^\delta} \|\nabla \dot g(t)\|\,dt\Big)^2 \le \\
&\le \|\nabla u_r^\delta\|^2 + \mathcal{H}^1(K_r^\delta) + 2\int_{t_r^\delta}^{t_{r+1}^\delta} (\nabla u_\delta(t)|\nabla \dot g(t))\,dt + \sigma(\delta)\int_{t_r^\delta}^{t_{r+1}^\delta} \|\nabla \dot g(t)\|\,dt\,,
\end{aligned}$$

where

$$\sigma(\delta) := \max_{i\le r<j} \int_{t_r^\delta}^{t_{r+1}^\delta} \|\nabla \dot g(t)\|\, dt \ \longrightarrow\ 0$$

by the absolute continuity of the integral. Iterating now this inequality for $i \le r < j$ we get (7.2) with $\rho(\delta) := \sigma(\delta)\int_0^1 \|\nabla \dot g(t)\|\, dt$. □

Lemma 7.4. *There exists a constant λ, depending only on g and K_0, such that*

$$\|\nabla u_i^\delta\| \le \lambda \quad \textit{and} \quad \mathcal{H}^1(K_i^\delta) \le \lambda \tag{7.6}$$

for every $\delta > 0$ and for every $0 \le i \le N_\delta$.

Proof. As g_i^δ is admissible for the problem (6.5) which defines $\mathcal{E}(g_i^\delta, K_i^\delta)$, by the minimality of u_i^δ we have $\|\nabla u_i^\delta\| \le \|\nabla g_i^\delta\|$, hence $\|\nabla u_\delta(t)\| \le \|\nabla g_\delta(t)\|$ for every $t \in [0,1]$. As $t \mapsto g(t)$ is absolutely continuous with values in $H^1(\Omega)$ the function $t \mapsto \|\nabla \dot g(t)\|$ is integrable on $[0,1]$ and there exists a constant $C > 0$ such that $\|\nabla g(t)\| \le C$ for every $t \in [0,1]$. This implies the former inequality in (7.6). The latter inequality follows now from Lemma 7.3 and from the inequality $\|\nabla u_0^\delta\|^2 + \mathcal{H}^1(K_0^\delta) \le \|\nabla g(0)\|^2 + \mathcal{H}^1(K_0)$, which is an obvious consequence of (7.1) for $i = 0$. □

Lemma 7.5. *Let λ be the constant of Lemma 7.4. There exists an increasing function $K\colon [0,1] \to \mathcal{K}_m^\lambda(\overline{\Omega})$ such that, for every $t \in [0,1]$, $K_\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$ along a suitable sequence independent of t.*

Proof. By Theorem 6.3 there exists an increasing function $K\colon [0,1] \to \mathcal{K}(\overline{\Omega})$ such that, for every $t \in [0,1]$, $K_\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$ along a suitable sequence independent of t. By Lemma 7.4 we have $\mathcal{H}^1(K_\delta(t)) \le \lambda$ for every $t \in [0,1]$ and every $\delta > 0$. By Corollary 3.3 this implies $K(t) \in \mathcal{K}_m^\lambda(\overline{\Omega})$ for every $t \in [0,1]$. □

In the rest of this section, when we write $\delta \to 0$, we always refer to the sequence given by Lemma 7.5.

For every $t \in [0,1]$ let $u(t)$ be a solution of the minimum problem (6.5) which defines $\mathcal{E}(g(t), K(t))$.

Lemma 7.6. *For every $t \in [0,1]$ we have $\nabla u_\delta(t) \to \nabla u(t)$ strongly in $L^2(\Omega;\mathbb{R}^2)$.*

Proof. As $u_\delta(t)$ is a solution of the minimum problem (6.5) which defines $\mathcal{E}(g_\delta(t), K_\delta(t))$, and $g_\delta(t) \to g(t)$ strongly in $H^1(\Omega)$, the conclusion follows from Theorem 5.1. □

Lemma 7.7. *For every $t \in [0,1]$ we have*

$$\mathcal{E}(g(t), K(t)) \le \mathcal{E}(g(t), K) \quad \forall\, K \in \mathcal{K}_m^f(\overline{\Omega}),\ K \supset K(t). \tag{7.7}$$

Moreover

$$\mathcal{E}(g(0), K(0)) \le \mathcal{E}(g(0), K) \quad \forall\, K \in \mathcal{K}_m^f(\overline{\Omega}),\ K \supset K_0. \tag{7.8}$$

Proof. Let us fix $t \in [0,1]$ and $K \in \mathcal{K}_m^f(\overline{\Omega})$ with $K \supset K(t)$. Since $K_\delta(t)$ converges to $K(t)$ in the Hausdorff metric as $\delta \to 0$, by Lemma 3.5 there exists a sequence (K_δ) in $\mathcal{K}_m^f(\overline{\Omega})$, converging to K in the Hausdorff metric, such that $K_\delta \supset K_\delta(t)$ and $\mathcal{H}^1(K_\delta \setminus K_\delta(t)) \to \mathcal{H}^1(K \setminus K(t))$ as $\delta \to 0$. By Lemma 7.4 this implies that $\mathcal{H}^1(K_\delta)$ is bounded as $\delta \to 0$.

Let v_δ and v be solutions of the minimum problems (6.5) which define $\mathcal{E}(g_\delta(t), K_\delta)$ and $\mathcal{E}(g(t), K)$, respectively. By Theorem 5.1 $\nabla v_\delta \to \nabla v$ strongly in $L^2(\Omega;\mathbb{R}^2)$. The minimality of $K_\delta(t)$ expressed by (7.1) gives $\mathcal{E}(g_\delta(t), K_\delta(t)) \le \mathcal{E}(g_\delta(t), K_\delta)$, which implies $\|\nabla u_\delta(t)\|^2 \le \|\nabla v_\delta\|^2 + \mathcal{H}^1(K_\delta \setminus K_\delta(t))$. Passing to the limit as $\delta \to 0$ and using Lemma 7.6 we get $\|\nabla u(t)\|^2 \le \|\nabla v\|^2 + \mathcal{H}^1(K \setminus K(t))$. Adding $\mathcal{H}^1(K(t))$ to both sides we obtain (7.7).

A similar proof holds for (7.8). By (7.1) we have $\mathcal{E}(g_\delta(0), K_\delta(0)) \le \mathcal{E}(g_\delta(0), K) = \mathcal{E}(g(0), K)$, which implies $\|\nabla u_\delta(0)\|^2 + \mathcal{H}^1(K_\delta(0)) \le \mathcal{E}(g(0), K)$. Passing to the limit as $\delta \to 0$ and using Lemma 7.6 and Corollary 3.3 we obtain (7.8). □

The previous lemma proves conditions (b) and (c) of Theorem 7.1. To show that conditions (d) and (e) are also satisfied, we begin by proving the following inequality.

Lemma 7.8. *For every* s, t *with* $0 \le s < t \le 1$

$$\|\nabla u(t)\|^2 + \mathcal{H}^1(K(t)) \le \|\nabla u(s)\|^2 + \mathcal{H}^1(K(s)) + 2\int_s^t (\nabla u(\tau)|\nabla \dot g(\tau))d\tau\,. \tag{7.9}$$

Proof. Let us fix s, t with $0 \le s < t \le 1$. Given $\delta > 0$ let i and j be the integers such that $t_i^\delta \le s < t_{i+1}^\delta$ and $t_j^\delta \le t < t_{j+1}^\delta$. Let us define $s_\delta := t_i^\delta$ and $t_\delta := t_j^\delta$. Applying Lemma 7.3 we obtain

$$\|\nabla u_\delta(t)\|^2 + \mathcal{H}^1(K_\delta(t)\backslash K_\delta(s)) \le \|\nabla u_\delta(s)\|^2 + 2\int_{s_\delta}^{t_\delta} (\nabla u_\delta(\tau)|\nabla \dot g(\tau))\,d\tau + \rho(\delta)\,, \tag{7.10}$$

with $\rho(\delta)$ converging to zero as $\delta \to 0$. By Lemma 7.6 for every $\tau \in [0,1]$ we have $\nabla u_\delta(\tau) \to \nabla u(\tau)$ strongly in $L^2(\Omega, \mathbb{R}^2)$ as $\delta \to 0$, and by Lemma 7.4 we have $\|\nabla u_\delta(\tau)\| \le \lambda$ for every $\tau \in [0,1]$. By Corollary 3.4 we get

$$\mathcal{H}^1(K(t)\backslash K(s)) \le \liminf_{\delta\to 0}\, \mathcal{H}^1(K_\delta(t)\backslash K_\delta(s))\,.$$

Passing now to the limit in (7.10) as $\delta \to 0$ we obtain (7.9). □

The following lemma concludes the proof of Theorem 7.1, showing that also conditions (d), (e) and (f) are satisfied.

Lemma 7.9. *The function* $t \mapsto \mathcal{E}(g(t), K(t))$ *is absolutely continuous on* $[0,1]$ *and*

$$\frac{d}{dt}\mathcal{E}(g(t), K(t)) = 2(\nabla u(t)|\nabla \dot g(t)) \qquad \text{for a.e. } t \in [0,1]\,. \tag{7.11}$$

Moreover

$$\frac{d}{ds}\mathcal{E}(g(t), K(s))\Big|_{s=t} = 0 \qquad \text{for a.e. } t \in [0,1]\,. \tag{7.12}$$

Proof. Let $0 \le s < t \le 1$. From the previous lemma we get

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s)) \le 2\int_s^t (\nabla u(\tau)|\nabla \dot g(\tau))\,d\tau\,. \tag{7.13}$$

On the other hand, by condition (c) of Theorem 7.1 we have $\mathcal{E}(g(s), K(s)) \le \mathcal{E}(g(s), K(t))$, and by Lemma 6.5

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(t)) = 2\int_s^t (\nabla u(\tau, t)|\nabla \dot g(\tau))\,d\tau\,,$$

where $u(\tau, t)$ is a solution of the minimum problem (6.5) which defines $\mathcal{E}(g(\tau), K(t))$. Therefore

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s)) \ge 2\int_s^t (\nabla u(\tau, t)|\nabla \dot g(\tau))\,d\tau\,. \tag{7.14}$$

Since there exists a constant C such that $\|\nabla u(\tau)\| \le \|\nabla g(\tau)\| \le C$ and $\|\nabla u(\tau, t)\| \le \|\nabla g(\tau)\| \le C$ for $s \le \tau \le t$, from (7.13) and (7.14) we obtain

$$\big|\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(s), K(s))\big| \le 2\,C\int_s^t \|\nabla \dot g(\tau)\|\,d\tau\,,$$

which proves that the function $t \mapsto \mathcal{E}(g(t), K(t))$ is absolutely continuous.

As $\nabla u(\tau, t) \to \nabla u(t)$ strongly in $L^2(\Omega, \mathbb{R}^2)$ when $\tau \to t$, if we divide (7.13) and (7.14) by $t - s$, and take the limit as $s \to t-$ we obtain (7.11). Equality (7.12) follows from Theorem 6.4. □

Theorem 1.1 is a consequence of Theorem 7.1 and of the following lemma.

Lemma 7.10. *Let $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ be a function which satisfies conditions (a)–(e) of Theorem 7.1. Then, for $0 < t \leq 1$,*

$$\mathcal{E}(g(t), K(t)) \leq \mathcal{E}(g(t), K) \quad \forall\, K \in \mathcal{K}_m^f(\overline{\Omega}) \;\; K \supset \textstyle\bigcup_{s<t} K(s)\,. \tag{7.15}$$

Proof. Let us fix t, with $0 < t \leq 1$, and $K \in \mathcal{K}_m^f(\overline{\Omega})$, with $K \supset \bigcup_{s<t} K(s)$. For $0 \leq s < t$ we have $K \supset K(s)$, and from condition (c) of Theorem 7.1 we obtain $\mathcal{E}(g(s), K(s)) \leq \mathcal{E}(g(s), K)$. As the functions $s \mapsto \mathcal{E}(g(s), K(s))$ and $s \mapsto \mathcal{E}(g(s), K)$ are continuous, passing to the limit as $s \to t-$ we get (7.15). □

The following lemma shows that $K(t)$, $K^-(t)$, and $K^+(t)$ have the same total energy.

Lemma 7.11. *Let $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ be a function which satisfies conditions (a)–(e) of Theorem 7.1, and let $K^-(t)$ and $K^+(t)$ be defined by (6.1) and (6.2). Then*

$$\mathcal{E}(g(t), K(t)) = \mathcal{E}(g(t), K^-(t)) \quad \textit{for } 0 < t \leq 1\,, \tag{7.16}$$

$$\mathcal{E}(g(t), K(t)) = \mathcal{E}(g(t), K^+(t)) \quad \textit{for } 0 \leq t < 1\,. \tag{7.17}$$

Proof. Let $0 < t \leq 1$. Since $K(s) \to K^-(t)$ in the Hausdorff metric as $s \to t-$, and $\mathcal{H}^1(K(s)) \to \mathcal{H}^1(K^-(t))$ by Corollary 3.3, it follows that $\mathcal{E}(g(s), K(s)) \to \mathcal{E}(g(t), K^-(t))$ as $s \to t-$ by Theorem 5.1. As the function $s \mapsto \mathcal{E}(g(s), K(s))$ is continuous, we obtain (7.16). The proof of (7.17) is analogous. □

Remark 7.12. From Lemmas 7.10 and 7.11 it follows that, if $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ is a function which satisfies conditions (a)–(e) of Theorem 7.1, the same is true for the functions

$$t \mapsto \begin{cases} K(0) & \text{for } t = 0\,, \\ K^-(t) & \text{for } 0 < t \leq 1\,, \end{cases} \qquad t \mapsto \begin{cases} K^+(t) & \text{for } 0 \leq t < 1\,, \\ K(1) & \text{for } t = 1\,, \end{cases}$$

where $K^-(t)$ and $K^+(t)$ are defined by (6.1) and (6.2). Therefore the problem has a left-continuous solution and a right-continuous solution.

Remark 7.13. In Theorem 7.1 suppose that $\mathcal{E}(g(0), K_0) \leq \mathcal{E}(g(0), K)$ for every $K \in \mathcal{K}_m^f(\overline{\Omega})$ with $K \supset K_0$. Then in our time discretization process we can take $K_0^\delta = K_0$ for every $\delta > 0$. Therefore there exists a function $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$, satisfying conditions (a)–(e) of Theorem 7.1, such that $K(0) = K_0$. In particular this happens for every K_0 whenever $g(0) = 0$.

In the case $g(0) = 0$, by condition (b) we must have $\mathcal{H}^1(K(0) \setminus K_0) = 0$, hence $K(0) = K_0$ if $K(0)$ has no isolated points. If we disregard this natural constraint and K_0 has m_0 connected components, for every finite set $F \subset \overline{\Omega}$ with no more than $m - m_0$ elements we can find also a solution with $K(0) = K_0 \cup F$. Indeed in our time discretization process we can take $K_0^\delta = K_0 \cup F$ for every $\delta > 0$.

We consider now the case where $g(t)$ is proportional to a fixed function $h \in H^1(\Omega)$.

Proposition 7.14. *In Theorem 7.1 suppose that $g(t) = \varphi(t)\, h$, where $\varphi \in AC([0,1])$ is non-decreasing and non-negative, and h is a fixed function in $H^1(\Omega)$. Let $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ be a function which satisfies conditions (a)–(e) of Theorem 7.1. Then*

$$\mathcal{E}(g(t), K(t)) \leq \mathcal{E}(g(t), K(s)) \tag{7.18}$$

for $0 \leq s < t \leq 1$.

Proof. Let us fix $0 \leq s < t \leq 1$. For every $\tau \in [0,1]$ let $v(\tau)$ be a solution of the minimum problem (6.5) which defines $\mathcal{E}(h, K(\tau))$. As $u(\tau) = \varphi(\tau)\, v(\tau)$ and $\dot g(\tau) = \dot\varphi(\tau)\, h$, from condition (f) we obtain, adding and subtracting $\mathcal{E}(g(s), K(s))$,

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(t), K(s)) =$$
$$= 2 \int_s^t (\nabla v(\tau) | \nabla h)\, \varphi(\tau)\, \dot\varphi(\tau)\, d\tau + (\varphi(s)^2 - \varphi(t)^2) \|\nabla v(s)\|^2\,.$$

As $v(\tau)$ is a solution of problem (4.2) with $K = K(\tau)$, and $v(\tau) = h$ q.e. on $\partial_D\Omega\backslash K(\tau)$, we have $(\nabla v(\tau)|\nabla h) = \|\nabla v(\tau)\|^2$. By the monotonicity of $\tau \mapsto K(\tau)$, for $s \le \tau \le t$ we have $v(s) \in L^{1,2}(\Omega\backslash K(\tau))$ and $v(s) = h$ q.e. on $\partial_D\Omega\backslash K(\tau)$. By the minimum property of $v(\tau)$ we obtain $\|\nabla v(\tau)\|^2 \le \|\nabla v(s)\|^2$ for $s \le \tau \le t$. Therefore

$$\mathcal{E}(g(t), K(t)) - \mathcal{E}(g(t), K(s)) \le$$
$$\le 2\int_s^t \varphi(\tau)\,\dot\varphi(\tau)\,d\tau\ \|\nabla v(s)\|^2 + (\varphi(s)^2 - \varphi(t)^2)\|\nabla v(s)\|^2 = 0\,,$$

which concludes the proof. □

8. BEHAVIOUR NEAR THE TIPS

In this section we consider a function $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ which satisfies conditions (a)–(e) of Theorem 7.1 for a suitable $g \in AC([0,1]; H^1(\Omega))$, and we study the behaviour of the solutions $u(t)$ near the "tips" of the sets $K(t)$. Under some natural assumptions, we shall see that $K(t)$ satisfies Griffith's criterion for crack growth.

For every bounded open set $A \subset \mathbb{R}^2$ with Lipschitz boundary, for every compact set $K \subset \mathbb{R}^2$, and for every function $g\colon \partial A\backslash K \to \mathbb{R}$ we define

$$\mathcal{E}(g, K, A) := \min_{v \in \mathcal{V}(g,K,A)} \Big\{ \int_{A\backslash K} |\nabla v|^2\,dx + \mathcal{H}^1(K \cap \overline{A}) \Big\}\,, \tag{8.1}$$

where

$$\mathcal{V}(g, K, A) := \{v \in L^{1,2}(A\backslash K) : v = g \quad \text{q.e. on } \partial A\backslash K\}\,.$$

We now consider in particular the case where K is a regular arc, and summarize some known results on the behaviour of a solution of problem (4.2) near the end-points of K. Let B be an open ball in $\mathbb{R}^2$ and let $\gamma\colon [\sigma_0, \sigma_1] \to \mathbb{R}^2$ be a simple path of class C^2 parametrized by arc length. Assume that $\gamma(\sigma_0) \in \partial B$ and $\gamma(\sigma_1) \in \partial B$, while $\gamma(\sigma) \in B$ for $\sigma_0 < \sigma < \sigma_1$. Assume in addition that γ is not tangent to ∂B at σ_0 and σ_1. For every $\sigma \in [\sigma_0, \sigma_1]$ let $\Gamma(\sigma) := \{\gamma(s) : \sigma_0 \le s \le \sigma\}$.

Theorem 8.1. *Let $\sigma_0 < \sigma < \sigma_1$ and let u be a solution to problem (4.2) with $\Omega = B$, $\partial_D\Omega = \partial B$, and $K = \Gamma(\sigma)$. Then there exists a unique constant $\kappa = \kappa(u, \sigma) \in \mathbb{R}$ such that*

$$u - \kappa\sqrt{2\rho/\pi}\,\sin(\theta/2) \in H^2(B\backslash\Gamma(\sigma)) \cap H^{1,\infty}(B\backslash\Gamma(\sigma))\,, \tag{8.2}$$

where $\rho(x) = |x - \gamma(\sigma)|$ and $\theta(x)$ is the continuous function on $B\backslash\Gamma(\sigma)$ which coincides with the oriented angle between $\dot\gamma(\sigma)$ and $x - \gamma(\sigma)$, and vanishes on the points of the form $x = \gamma(\sigma) + \varepsilon\,\dot\gamma(\sigma)$ for sufficiently small $\varepsilon > 0$.

Proof. Let B^- and B^+ be the connected components of $B\backslash\Gamma(\sigma_1)$. Since B^- and B^+ have a Lipschitz boundary, by Proposition 2.2 u belongs to $H^1(B^-)$ and $H^1(B^+)$. This implies that $u \in L^2(B)$, and hence $u \in H^1(B\backslash\Gamma(\sigma))$. The conclusion follows now from [23, Theorem 4.4.3.7 and Section 5.2], as shown in [30, Appendix 1]. □

Remark 8.2. If u is interpreted as the third component of the displacement in an anti-plane shear, as we did in the introduction, then κ coincides with the *Mode III stress intensity factor* K_{III} of the displacement $(0, 0, u)$.

Theorem 8.3. *Let $g\colon \partial B\backslash\{\gamma(\sigma_0)\} \to \mathbb{R}$ be a function such that for every $\sigma_0 < \sigma < \sigma_1$ there exists $g(\sigma) \in L^{1,2}(B\backslash\Gamma(\sigma))$ with $g(\sigma) = g$ q.e. on $\partial B\backslash\Gamma(\sigma) = \partial B\backslash\{\gamma(\sigma_0)\}$. Let $v(\sigma)$ be a solution of the minimum problem (8.1) which defines $\mathcal{E}(g, \Gamma(\sigma), B)$. Then, for every $\sigma_0 < \sigma < \sigma_1$,*

$$\frac{d}{d\sigma}\mathcal{E}(g, \Gamma(\sigma), B) = 1 - \kappa(v(\sigma), \sigma)^2\,,$$

where κ is defined by (8.2).

Proof. It is enough to adapt the proof of [24, Theorem 6.4.1]. □

Let us return to the function $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ considered at the beginning of the section, and let $0 \le t_0 < t_1 \le 1$. Suppose that the following structure condition is satisfied: there exist a finite family of simple arcs Γ_i, $i = 1, \dots, p$, contained in Ω and parametrized by arc length by C^2 paths $\gamma_i\colon [\sigma_i^0, \sigma_i^1] \to \Omega$, such that, for $t_0 < t < t_1$,

$$K(t) = K(t_0) \cup \bigcup_{i=1}^{p} \Gamma_i(\sigma_i(t)), \tag{8.3}$$

where $\Gamma_i(\sigma) := \{\gamma_i(\tau) : \sigma_i^0 \le \tau \le \sigma\}$ and $\sigma_i\colon [t_0, t_1] \to [\sigma_i^0, \sigma_i^1]$ are non-decreasing functions with $\sigma_i(t_0) = \sigma_i^0$ and $\sigma_i^0 < \sigma_i(t) < \sigma_i^1$ for $t_0 < t < t_1$. Assume also that the arcs Γ_i are pairwise disjoint, and that $\Gamma_i \cap K(t_0) = \{\gamma_i(\sigma_i^0)\}$. We consider the sets $\Gamma_i(\sigma_i(t))$ as the increasing branches of the fracture $K(t)$ and the points $\gamma_i(\sigma_i(t))$ as their moving tips. For $i = 1, \dots, p$ and $\sigma_i^0 < \sigma < \sigma_i^1$ let $\kappa_i(u, \sigma)$ be the stress intensity factor defined by (8.2) with $\gamma = \gamma_i$ and B equal to a sufficiently small ball centred at $\gamma_i(\sigma)$.

We are now in a position to state the main result of this section.

Theorem 8.4. *Let $m \ge 1$, let $K\colon [0,1] \to \mathcal{K}_m^f(\overline{\Omega})$ be a function which satisfies conditions (a)–(e) of Theorem 7.1 for a suitable $g \in AC([0,1]; H^1(\Omega))$, let $u(t)$ be a solution of the minimum problem (6.5) which defines $\mathcal{E}(g(t), K(t))$, and let $0 \le t_0 < t_1 \le 1$. Assume that (8.3) is satisfied for $t_0 < t < t_1$, and that the arcs Γ_i and the functions σ_i satisfy all properties considered above. Then*

$$\dot\sigma_i(t) \ge 0 \quad \text{for a.e. } t \in (t_0, t_1), \tag{8.4}$$

$$1 - \kappa_i(u(t), \sigma_i(t))^2 \ge 0 \quad \text{for every } t \in (t_0, t_1), \tag{8.5}$$

$$\{1 - \kappa_i(u(t), \sigma_i(t))^2\}\, \dot\sigma_i(t) = 0 \quad \text{for a.e. } t \in (t_0, t_1), \tag{8.6}$$

for $i = 1, \dots, p$.

The first condition says simply that the length of every branch of the fracture can not decrease, and reflects the irreversibility of the process. The second condition says that the absolute value of the stress intensity factor must be less than or equal to 1 at each tip and for every time. The last condition says that, at a given tip, the stress intensity factor must be equal to ± 1 at almost every time in which this tip moves with a positive velocity. This is Griffith's criterion for crack growth in our model.

To prove Theorem 8.4 we use the following lemma.

Lemma 8.5. *Let $m \ge 1$, let $H \in \mathcal{K}_m^f(\overline{\Omega})$ with h connected components, let $g \in H^1(\Omega)$, and let u be a solution of the minimum problem (6.5) which defines $\mathcal{E}(g, H)$. Given an open subset A of Ω, with Lipschitz boundary, such that $H \cap \overline{A} \ne \emptyset$, let q be the number of connected components of H which meet $\overline{A}$. Assume that*

$$\mathcal{E}(g, H) \le \mathcal{E}(g, K) \qquad \forall\, K \in \mathcal{K}_m^f(\overline{\Omega}),\ \ K \supset H. \tag{8.7}$$

Then

$$\mathcal{E}(u, H, A) \le \mathcal{E}(u, K, A) \qquad \forall\, K \in \mathcal{K}_{q+m-h}^f(\overline{A}),\ \ K \supset H \cap \overline{A}. \tag{8.8}$$

Proof. Let $K \in \mathcal{K}_{q+m-h}^f(\overline{A})$ with $K \supset H \cap \overline{A}$, let v be a solution of the minimum problem (8.1) which defines $\mathcal{E}(u, K, A)$, and let w be the function defined by $w := v$ on $\overline{A} \backslash K$ and by $w := u$ on $(\overline{\Omega} \backslash \overline{A}) \backslash H$. As $v = u$ q.e. on $\partial A \backslash K$ the function w belongs to $L^{1,2}(\Omega \backslash (H \cup K))$; using also the fact that $u = g$ q.e. on $\partial_D \Omega \backslash H$, we obtain that $w = g$ q.e. on $\partial_D \Omega \backslash (H \cup K)$. Therefore

$$\begin{aligned} \mathcal{E}(g, H \cup K) &\le \int_{\Omega \backslash (H \cup K)} |\nabla w|^2\, dx + \mathcal{H}^1(H \cup K) = \\ &= \int_{A \backslash K} |\nabla v|^2\, dx + \mathcal{H}^1(K \cap \overline{A}) + \int_{(\Omega \backslash A) \backslash H} |\nabla u|^2\, dx + \mathcal{H}^1(H \backslash \overline{A}). \end{aligned} \tag{8.9}$$

On the other hand, by the minimality of u,

$$\int_{A\setminus H} |\nabla u|^2\,dx + \mathcal{H}^1(H\cap\overline{A}) + \int_{(\Omega\setminus A)\setminus H} |\nabla u|^2\,dx + \mathcal{H}^1(H\setminus\overline{A}) = \tag{8.10}$$

$$= \int_{\Omega\setminus H} |\nabla u|^2\,dx + \mathcal{H}^1(H) = \mathcal{E}(g,H) \le \mathcal{E}(g, H\cup K)\,,$$

where the last inequality follows from (8.7), since $H\cup K$ has no more than m connected components (indeed, $H\cup K$ has exactly $h-q$ connected components which do not meet $\overline{A}$, and every connected component of $H\cup K$ which meets $\overline{A}$ contains a connected component of K, so that their number does not exceed $q+m-h$). From (8.9) and (8.10) we obtain

$$\int_{A\setminus H} |\nabla u|^2\,dx + \mathcal{H}^1(H\cap\overline{A}) \le \int_{A\setminus K} |\nabla v|^2\,dx + \mathcal{H}^1(K\cap\overline{A})\,,$$

and the minimality of v yields (8.8). ☐

Proof of Theorem 8.4. Let t be an arbitrary point in (t_0,t_1) and let B_i, $i=1,\dots,p$, be a family of open balls centred at the points $\gamma_i(\sigma_i(t))$. If the radii are sufficiently small, we have $\overline{B}_i\subset\Omega$ and $\overline{B}_i\cap K(t_0) = \overline{B}_i\cap\overline{B}_j = \overline{B}_i\cap\Gamma_j = \emptyset$ for $j\ne i$. Moreover we may assume that $B_i\cap\Gamma_i = \{\gamma_i(\sigma) : \tau_i^0 < \sigma < \tau_i^1\}$, for suitable constants τ_i^0, τ_i^1 with $\sigma_i^0 < \tau_i^0 < \sigma_i(t) < \tau_i^1 < \sigma_i^1$, and that the arcs Γ_i intersect ∂B_i only at the points $\gamma_i(\tau_i^0)$ and $\gamma_i(\tau_i^1)$, with a transversal intersection. All these properties, together with (8.3), imply that

$$\overline{B}_i\cap K(s) = \overline{B}_i\cap\Gamma_i(\sigma_i(s)) = \{\gamma_i(\sigma) : \tau_i^0\le\sigma\le\sigma_i(s)\} \quad\text{if}\quad \tau_i^0 < \sigma_i(s) < \tau_i^1\,. \tag{8.11}$$

In particular this happens for $s=t$, and for s close to t if σ_i is continuous at t.

By condition (c) of Theorem 7.1 and by Lemma 8.5 for every i we have that

$$\mathcal{E}(u(t),K(t),B_i) \le \mathcal{E}(u(t),K,B_i) \quad \forall K\in\mathcal{K}_1^f(\overline{B}_i),\ \ K\supset K(t)\cap\overline{B}_i\,.$$

By (8.11) this implies, taking $K := \Gamma_i(\sigma)\cap\overline{B}_i = \{\gamma_i(\tau) : \tau_i^0\le\tau\le\sigma\}$,

$$\mathcal{E}(u(t),\Gamma_i(\sigma_i(t)),B_i) \le \mathcal{E}(u(t),\Gamma_i(\sigma),B_i) \qquad\text{for } \sigma_i(t)\le\sigma\le\tau_i^1\,,$$

which yields

$$\frac{d}{d\sigma}\mathcal{E}(u(t),\Gamma_i(\sigma),B_i)\Big|_{\sigma=\sigma_i(t)} \ge 0\,. \tag{8.12}$$

Inequality (8.5) follows now from Theorem 8.3 applied with $g:=u(t)$.

By condition (e) of Theorem 7.1 for a.e. in $t\in(t_0,t_1)$ we have $\frac{d}{ds}\mathcal{E}(g(t),K(s))|_{s=t}=0$. Moreover, for a.e. in $t\in(t_0,t_1)$ the derivative $\dot\sigma_i(t)$ exists for $i=1,\dots,p$. Let us fix $t\in(t_0,t_1)$ which satisfies all these properties.

By (8.11) for s close to t we have

$$\mathcal{E}(g(t),K(s)) \le \sum_{i=1}^p \mathcal{E}(u(t),\Gamma_i(\sigma_i(s)),B_i) + \mathcal{E}(u(t),K,A)\,, \tag{8.13}$$

where $K := K(t_0)\cup\bigcup_i\Gamma_i(\tau_i^0)$ and $A := \Omega\setminus\bigcup_i\overline{B}_i$. Note that the equality holds in (8.13) for $s=t$. As the functions $s\mapsto\mathcal{E}(g(t),K(s))$ and $s\mapsto\mathcal{E}(u(t),\Gamma_i(\sigma_i(s)),B_i)$ are differentiable at $s=t$ (by Theorem 8.3 and by the existence of $\dot\sigma_i(t)$), we conclude that

$$0 = \frac{d}{ds}\mathcal{E}(g(t),K(s))\Big|_{s=t} = \sum_{i=1}^p \frac{d}{ds}\mathcal{E}(u(t),\Gamma_i(\sigma_i(s)),B_i)\Big|_{s=t} =$$

$$= \sum_{i=1}^p \frac{d}{d\sigma}\mathcal{E}(u(t),\Gamma_i(\sigma),B_i)\Big|_{\sigma=\sigma_i(t)}\dot\sigma_i(t) = \sum_{i=1}^p \big\{1-\kappa_i(u(t),\sigma_i(t))^2\big\}\,\dot\sigma_i(t)\,.$$

By (8.4) and (8.5) we have $\left\{1 - \kappa_i(u(t), \sigma_i(t))^2\right\} \dot{\sigma}_i(t) \geq 0$ for $i = 1, \ldots, p$, so that the previous equalities yield (8.6). $\square$

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